Name:

Quiz 6

Part I (10 points). You will get 1 point for each correct answer, 0 points for each blank answer, and -1 point for each incorrect answer. The minimum possible score for this section is 0.

In all of the following, V is a finite dimensional complex vector space and $T \in \mathcal{L}(V)$. We write p_{char} and $p_{\min}$ for the characteristic and minimal polynomials of T, respectively.

(1) The degree of $p_{\min}$ is the number of distinct eigenvalues of T. T **F**

(2) If $\dim V = 3$ and T has eigenvalues 0, 2 and -2, then $T^3 = 4T$. **T** F

(3) If $\dim V = 5$ and $\operatorname{null} T \neq \operatorname{null} T^2$, then T has at most 3 distinct eigenvalues. T **F**

(4) T is not surjective if and only if $p_{\text{char}}(0) = 0$. **T** F

(5) If the matrix representation of T with respect to some basis is **T** F

$$M(T) = \begin{pmatrix} 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

then $p_{\text{char}} = p_{\min}$.

(6) Suppose 2 is the only eigenvalue of T and that we know the following. T **F**

$$\begin{aligned} \dim \operatorname{null}(T-2I) &= 2 \\ \dim \operatorname{null}(T-2I)^2 &= 4 \\ \dim \operatorname{null}(T-2I)^3 &= 6 \\ \dim \operatorname{null}(T-2I)^4 &= 6 \end{aligned}$$

Then $p_{\min}(z) = (z-2)^2$.

(7) If the matrix of T with respect to some basis is in Jordan form with two Jordan blocks of eigenvalue 0, one of size 2 and another one of size 3, then $\dim \operatorname{null} T^2 = \dim \operatorname{null} T + 2$. **T** F

(8) If $p_{\text{char}}(z) = (z-3)^2(z-2)^5$, it is possible to have $p_{\min}(z) = (z-3)^4(z-2)$. T **F**

(9) If $T^2 = T$, then T is diagonalizable with all eigenvalues equal to 0 or 1. **T** F

(10) If 7 is an eigenvalue of T and $\operatorname{null}(T-7I)^3 = \operatorname{null}(T-7I)^4$, then 7 has multiplicity at most 3 as a root of $p_{\min}$. **T** F

Part II (10 points).

(11) Suppose $T \in \mathcal{L}(\mathbf{C}^4)$ is given by $T(w, x, y, z) = (z, 0, x, y)$. Let $v = (0, 1, 0, 0)$ and note that $T^4 v = 0$ but $T^3 v \neq 0$. Calculate a basis for V with respect to which the matrix of T is in Jordan form.

We generate a chain using v. We have $Tv = (0, 0, 1, 0)$, $T^2 v = (0, 0, 0, 1)$, $T^3 v = (1, 0, 0, 0)$ and $T^4 v = 0$. Thus the matrix of T with respect to $T^3 v, T^2 v, Tv, v$ is

$$M(T) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so this is a Jordan basis.

(12) Does there exist an operator T on a finite dimensional complex vector space V such that $\dim \operatorname{null} T = 1$ and $\dim \operatorname{null} T^2 = 3$? If so, provide an example and prove that it is an example. Otherwise, prove that it is impossible.

There cannot exist such an operator. Suppose for a contradiction that there did exist such an operator. Let $U = \operatorname{null} T$ and $W = \operatorname{null} T^2$. Observe that $\dim(W/U) = 3 - 1 = 2$. Choose a basis $w_1 + U, w_2 + U$ for the quotient space W/U. Since $w_i \in W = \operatorname{null} T^2$, we must have $Tw_i \in \operatorname{null} T = U$. But $\dim U = 1$, so there must exist constants a_1, a_2 not both equal to 0 such that

$$a_1 T w_1 + a_2 T w_2 = 0.$$

Then $a_1 w_1 + a_2 w_2 \in \operatorname{null} T = U$, which means that

$$a_1(w_1 + U) + a_2(w_2 + U) = 0$$

in W/U. This contradicts our choice of $w_1 + U, w_2 + U$ as a basis of W/U.