Name:

Quiz 5

Part I (10 points). You will get 1 point for each correct answer, 0 points for each blank answer, and -1 point for each incorrect answer. The minimum possible score for this section is 0.

(1)	There exists a unique polynomial of degree at most 3 whose graph passes through the points $(0,1)$, $(1,-1)$, $(2,-1)$, and $(3,2)$.	**T**	F
(2)	Suppose $T \in \mathcal{L}(\mathbf{F}^5)$ is not surjective and $\dim E(2,T) = 4$. Then T is diagonalizable.	**T**	F
(3)	Suppose $T \in \mathcal{L}(\mathbf{F}^3)$ has eigenvectors u, v, w with the property that none of these three vectors is a scalar multiple of either of the others. Then T is diagonalizable.	T	**F**
(4)	Suppose $T \in \mathcal{L}(V)$, U is an invariant subspace, and W is a subspace such that $V = U \oplus W$. Then W is invariant.	T	**F**
(5)	Suppose $T \in \mathcal{L}(\mathbf{F}^5)$ and $\dim E(4,T) = 4$. Then either $T - 2I$ or $T + 2I$ is invertible.	**T**	F
(6)	Suppose $T \in \mathcal{L}(\mathbf{F}^3)$ has two distinct invariant subspaces of dimension 2. Then there exists an invariant subspace of dimension 1.	**T**	F
(7)	Suppose $T \in \mathcal{L}(V)$ has no eigenvalues. Then T^2 has no eigenvalues.	T	**F**
(8)	Suppose $T \in \mathcal{L}(\mathbf{F}^3)$ and u, v, w are linearly independent eigenvectors. If $u + v, v + w, w$ are also eigenvectors, then T is a scalar multiple of the identity.	**T**	F
(9)	Suppose $T \in \mathcal{L}(\mathbf{F}^6)$ and U is a 5 dimensional invariant subspace such that the restriction operator $T\|_U$ is diagonalizable. Then T is diagonalizable.	T	**F**
(10)	Suppose 2 is an eigenvalue of $T \in \mathcal{L}(\mathbf{F}^5)$ such that the eigenspace $U = E(2,T)$ has dimension 2 and the quotient operator T/U has eigenvalues 4, 5, 6. Then T is diagonalizable.	**T**	F

Part II (10 points).

(11) Let $T \in \mathcal{L}(\mathbf{F}^2)$ be given by $T(x, y) = (x + y, 3y)$. Is T diagonalizable? If so, find a basis of $\mathbf{F}^2$ such that $M(T)$ is diagonal. Otherwise, explain why no such basis exists.

Observe that the matrix of T with respect to the standard basis is

$$\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$$

which happens to be upper triangular, so the eigenvalues of T are the diagonal entries 1 and 3. Since T has 2 distinct eigenvalues, it must be diagonalizable. Moreover, $e_1 = (1, 0)$ is an eigenvector with eigenvalue 1.

To find an eigenvector with eigenvalue 3, we calculate $E(3, T) = \operatorname{null}(T - 3I)$. Observe that, with respect to the standard basis, we have

$$M(T - 3I) = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

which means that $E(3, T) = \{(x, 2x) : x \in \mathbf{F}\} = \operatorname{span}((1, 2))$. In particular, $(1, 2)$ is an eigenvector with eigenvalue 3, so the matrix of T with respect to $(1, 0), (1, 2)$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix},$$

which is a diagonal matrix.

(12) Let V be a finite dimensional complex vector space and suppose $T \in \mathcal{L}(V)$ is an operator such that 4 is an eigenvalue of T^2. Show that

$$\dim \operatorname{null}(T - 2I) + \dim \operatorname{null}(T + 2I) > 0.$$

There are many approaches. Here is one approach that works when V is any vector space over any field (not necessarily finite dimensional, not necessarily $\mathbf{C}$). Let v be an eigenvector for T^2 with eigenvalue 4. Let $p(z) = z^2 - 4 = (z + 2)(z - 2)$ and observe that

$$0 = (T^2 - 4I)v = p(T)v = (T + 2I)(T - 2I)v.$$

Then either $(T - 2I)v = 0$, in which case v is an eigenvector of T with eigenvalue 2, or else $(T - 2I)v \neq 0$, in which case $(T + 2I)(T - 2I)v = 0$ means that $(T - 2I)v$ is an eigenvector of T with eigenvalue -2. Thus, either 2 or -2 must be an eigenvalue of T, so either $\dim E(2, T) > 0$ or $\dim E(-2, T) > 0$, so in either case

$$\dim \operatorname{null}(T - 2I) + \dim \operatorname{null}(T + 2I) = \dim E(2, T) + \dim E(-2, T) > 0.$$

Here is another approach that takes advantage of the fact that we have assumed that we are working with a finite dimensional complex vector space. We know that there exists a basis $v_1, \ldots, v_n$ for V such that $M(T)$ is upper triangular:

$$M(T) = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Then

$$M(T^2) = M(T)^2 = \begin{pmatrix} \lambda_1^2 & & * \\ & \ddots & \\ 0 & & \lambda_n^2 \end{pmatrix}.$$

Since 4 is an eigenvalue of T^2 and $M(T^2)$ is upper triangular, one of the diagonal entries of this matrix must be 4. In other words, there must exist some i such that $\lambda_i^2 = 4$. Then $\lambda_i = \pm 2$, so again we see that either 2 or -2 must be an eigenvalue of T. We can then finish the proof just like at the end of the first argument above.