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Quiz 2

Part I (10 points). You will get 1 point for each correct answer, 0 points for each blank answer, and -1 point for each incorrect answer. The minimum possible score for this section is 0.

(1)	Every list of 5 polynomials in $\mathcal{P}_3(\mathbf{F})$ is linearly dependent.	**T**	F
(2)	Every basis for $\mathcal{P}_3(\mathbf{F})$ contains a degree 2 polynomial.	T	**F**
(3)	Suppose u, v, w is a basis for a vector space V and $w' \in V$. Then u, v, w' is a basis for V if and only if w' is a nonzero scalar multiple of w.	T	**F**
(4)	If u, v, w and u', v', w' are both bases for a 3 dimensional vector space V, then $u + u', v + v', w + w'$ is a basis.	T	**F**
(5)	Suppose p_0, p_1, p_2, p_3 is a list of four polynomials in $\mathcal{P}_3(\mathbf{R})$ with the property that $p_i(3) = 0$ for each i. Then the list p_0, p_1, p_2, p_3 is linearly dependent.	**T**	F
(6)	If v_1, v_2, v_3, v_4 is a basis for V and U is a subspace of V such that $v_1, v_2 \in U$ but $v_3, v_4 \notin U$, then v_1, v_2 is a basis of U.	T	**F**
(7)	If U_1, U_2 and U_3 are subspaces of a finite dimensional vector space V, then $\dim(U_1 + U_2 + U_3) = \dim U_1 + \dim U_2 + \dim U_3$.	T	**F**
(8)	If U_1, U_2 and W are subspaces of a vector space V such that $U_1 \oplus W = V$ and $U_2 \oplus W = V$, then $U_1 = U_2$.	T	**F**
(9)	If $v_1, \ldots, v_m$ is linearly independent and $w \in V$ is a vector such that $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_m)$.	**T**	F
(10)	If V is a 3 dimensional vector space over $\mathbf{C}$, then it is 6 dimensional when it is regarded as a vector space over $\mathbf{R}$.	**T**	F

Part II (10 points). Compute dimensions of each of the following vector spaces. Justify your answers with complete proofs.

(11) $V = \{(x, y, z) \in \mathbf{R}^3 : x - 2y - 3z = 0\}$.

For any $(x, y, z) \in V$, observe that

$$(x, y, z) = (2y + 3z, y, z) = y(2, 1, 0) + z(3, 0, 1)$$

so $(2, 1, 0), (3, 0, 1)$ spans V. These vectors are clearly linearly independent since they are not scalar multiples of each other, so they form a basis of V. Thus $\dim V = 2$.

(12) $V = \left\{p \in \mathcal{P}_3(\mathbf{R}) : \int_{-2}^{2} p(x)\, dx = 0\right\}$.

Since x and x^3 are odd functions, clearly they are in V. Also, the function

$$c \mapsto \int_{-2}^{2} (x^2 + c)\, dx$$

is clearly continuous, has a positive value when $c = 0$, and has limit $-\infty$ as $c \to -\infty$, so the intermediate value theorem guarantees that there exists some c_0 such that

$$\int_{-2}^{2} (x^2 + c_0)\, dx = 0.$$

In other words, we have

$$x, x^3, x^2 + c_0$$

in V. These are polynomials of different degrees, so the list is linearly independent. Thus $\dim V \geq 3$.

Since V is a subspace of $\mathcal{P}_3(\mathbf{R})$, we also know that $\dim V \leq 4$. But clearly $V \neq \mathcal{P}_3(\mathbf{R})$, since for example 1 is a polynomial in $\mathcal{P}_3(\mathbf{R})$ that is not in V, so $\dim V \neq 4$, so we must have $\dim V = 3$.