Name:

Summer 2017 — Math 110
Final Exam

Section	**Max**	**Score**
Part I — True/False	15	
Part II — Examples	10	
Part III — Calculations	20	
Part IV — Proofs	10	
Total	55	

Part I — True/False. You will get 1 point for each correct answer, 0 points for each blank answer, and -1 point for each incorrect answer. The minimum possible score for this section is 0.

(1) Let $V = \{(a, b) : a, b \in \mathbf{R}\}$. Define addition on V coordinate-wise, and define a scalar multiplication operation @ by the formula T **F**

$$\lambda@(a, b) = (0, \lambda b)$$

for all $(a, b) \in V$ and $\lambda \in \mathbf{R}$. Then V, equipped with these operations, is a vector space over $\mathbf{R}$.

(2) The set $\{(a, b, c) \in \mathbf{R}^3 : a^3 = b^3\}$ is a subspace of $\mathbf{R}^3$. **T** F

(3) Suppose p_0, p_1, p_2 is a list of three polynomials in $\mathcal{P}_2(\mathbf{R})$ with the property that $p_i(1) = 0$ for each i. Then the list p_0, p_1, p_2 is linearly dependent. **T** F

(4) Suppose U is a subspace of a vector space V and define a map $T : V \to V$ by T **F**

$$T(v) = \begin{cases} v & \text{if } v \in U \\ 0 & \text{if } v \notin U. \end{cases}$$

Then T is linear.

(5) Every basis for $\mathcal{P}_3(\mathbf{F})$ contains a degree 2 polynomial. T **F**

(6) Suppose v_1, v_2, v_3 is a basis for V and U is a subspace of V such that $v_1 \in U$ but $v_2, v_3 \notin U$. Then $U = \operatorname{span}(v_1)$. T **F**

(7) If V and W are vector spaces, $T \in \mathcal{L}(V, W)$, and $v_1, \ldots, v_n$ is a list in V such that $Tv_1, \ldots, Tv_n$ is linearly independent, then $v_1, \ldots, v_n$ is linearly independent. **T** F

(8) Suppose V is a finite dimensional inner product space. Every self-adjoint $T \in \mathcal{L}(V)$ has a cube root. **T** F

(9) There exists $T \in \mathcal{L}(\mathbf{R}^4, \mathbf{R}^2)$ such that $\operatorname{null} T = \{(0, 0, 0, x) : x \in \mathbf{R}\}$. T **F**

(10) Suppose W_1, W_2 and U are all subspaces of a vector space V such that $U \oplus W_1 = V$ and $U \oplus W_2 = V$. Then $W_1 = W_2$. T **F**

(11) Suppose V is a finite dimensional inner product space, $U = \operatorname{span}(u_1, u_2, u_3)$ is a subspace, and $v \in V$ is a vector such that **T** F

$$|\langle v, u_1\rangle|^2 + |\langle v, u_2\rangle|^2 + |\langle v, u_3\rangle|^2 = 0.$$

Then $v \in E(1, P_{U^\perp})$.

(12) Suppose $T \in \mathcal{L}(\mathbf{R}^4)$ is not injective and $\dim E(9, T) = 3$. Then T is diagonalizable. **T** F

(13) Suppose $T \in \mathcal{L}(\mathbf{C}^4)$ satisfies $(T - 3I)(T - 2I)(T - I) = 0$. Then the minimal polynomial $p_{\min}$ of T is $p_{\min}(z) = (z - 3)(z - 2)(z - 1)$. T **F**

(14) Suppose V is a vector space, $T \in \mathcal{L}(V)$ is an operator, and $v \in V$ is a nonzero vector such that $(T - 3I)(T + 3I)v = 0$. Then v is an eigenvector of T, with eigenvalue either 3 or -3. T **F**

(15) Suppose V is the inner product space of continuous functions $[-1, 1] \to \mathbf{R}$, with inner product given by **T** F

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

Let $h \in V$ be given by $h(x) = |x|$. Then $P_{\operatorname{span}(1,x,x^2)}(h) = P_{\operatorname{span}(1,x^2)}(h)$.

Part II — Examples. For each of the following, give an example of an operator $T \in \mathcal{L}(\mathbf{C}^4)$ that has the required properties by writing down a matrix *in Jordan form* that represents the operator, if it is possible. If it is impossible, write "impossible." A correct answer is worth 2 points, a blank answer is 0.5 points, and an incorrect answer is 0 points. No justification is required.

(1) T has precisely 3 distinct eigenvalues and its minimal polynomial is of degree 3.

Any diagonal matrix with 3 distinct numbers along the diagonal (so one number must be repeated twice). Eg,

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 3 \end{pmatrix}$$

(2) T is nilpotent and there does *not* exist a basis of $\mathbf{C}^4$ of the form T^3v, T^2v, Tv, v for some $v \in \mathbf{C}^4$.

Any nilpotent matrix in Jordan form, except for the one that has only a single Jordan block of size 4. Eg,

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

(3) $\dim \operatorname{null}(T-2I) = 2$, $\dim \operatorname{null}(T-2I)^2 = 3$ and $\dim \operatorname{null}(T-2I)^3 = 4$.

A matrix with 2 Jordan blocks of eigenvalue 2, one of size 3 and another of size 1. Eg

$$\begin{pmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ & & & 2 \end{pmatrix}$$

(4) There is no 3 dimensional subspace of $\mathbf{C}^4$ invariant under T.

Impossible. If v_1, v_2, v_3, v_4 is a Jordan basis, then $\operatorname{span}(v_1, v_2, v_3)$ will always be a 3 dimensional invariant subspace.

(5) The quotient operator T/U is not injective, where $U = \operatorname{null} T$.

Any matrix which has a Jordan block with eigenvalue 0 of size at least 2. Eg, if v_1, v_2, v_3, v_4 is a Jordan basis with

$$M(T) = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix},$$

then $U = \operatorname{null} T = \operatorname{span}(v_1, v_2, v_3)$, then $\mathbf{C}^4/U = \operatorname{span}(v_4 + U)$ and

$$(T/U)(v_4 + U) = T(v_4) + U = v_3 + U = 0_{\mathbf{C}^3/U},$$

so T/U is the zero operator on the 1 dimensional vector space $\mathbf{C}^4/U$, which is clearly not injective.

Part III — Calculations. In each of the following, you are asked to calculate the dimension of a vector space. Write the dimension inside the box on the right. If the dimension is infinite, write "∞." A correct answer is worth 2 points, a blank answer is 0.5 points, and an incorrect answer is 0 points. No justification is required.

(1) Let $U = \{(x, y, z) \in \mathbf{R}^3 : x + 2y + 3z = 0\}$. Calculate $\dim(\mathbf{R}^3/U)$.	1
(2) Calculate $\dim \mathcal{L}(\mathcal{P}_2(\mathbf{R})) \times \mathcal{P}_3(\mathbf{R})'$.	13
(3) Let $T \in \mathcal{L}(\mathcal{P}_4(\mathbf{R}), \mathcal{P}_3(\mathbf{R}))$ be defined by $T(f)(z) = zf''(z)$. Calculate $\dim \operatorname{range} T$.	3
(4) Suppose $T \in \mathcal{L}(\mathbf{C}^4)$ has 4 distinct eigenvalues and $p_{\text{char}}(0) = 0$. Calculate $\dim \operatorname{range} T$.	3
(5) Suppose U is the space of self-adjoint operators inside $\mathcal{L}(\mathbf{R}^3)$. Calculate $\dim U$.	6
(6) Suppose $T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^3)$ and $\dim \operatorname{null} T = 2$. Calculate $\dim \operatorname{null} T'$.	0
(7) Suppose $T \in \mathcal{L}(\mathbf{F}^\infty)$ is the forward shift operator $T(x_0, x_1, x_2, \dots) = (0, x_0, x_1, \dots)$ and define $U = \operatorname{range} T$. Calculate $\dim \mathbf{F}^\infty/U$.	1
(8) Suppose V is the vector space of infinitely differentiable functions $\mathbf{R} \to \mathbf{R}$ and $T \in \mathcal{L}(V)$ is the operator $T(f) = f' - f$. Calculate $\dim \operatorname{null} T$.	1
(9) Suppose $S, T \in \mathcal{L}(\mathbf{R}^3, \mathbf{R})$ are nonzero linear maps with distinct null spaces. Calculate $\dim(\operatorname{null} S) \cap (\operatorname{null} T)$.	1
(10) Regard $\mathbf{C}^2$ as a *real* vector space and $\mathbf{R}^2$ as a subspace of $\mathbf{C}^2$. Calculate $\dim(\mathbf{C}^2/\mathbf{R}^2)$.	2

Part IV — Proofs. You must write rigorous arguments using complete sentences for full credit. Each problem is worth a maximum of 5 points.

(1) For an arbitrary vector space V, prove that $\mathcal{L}(\mathbf{F}^2, V)$ is isomorphic to $V \times V$.

Define $\Gamma : \mathcal{L}(\mathbf{F}^2, V) \to V \times V$ by declaring $\Gamma(T) = (Te_1, Te_2)$. If $S, T \in \mathcal{L}(\mathbf{F}^2, V)$ and $\lambda \in \mathbf{F}$, then

$$\Gamma(S + \lambda T) = ((S + \lambda T)e_1, (S + \lambda T)e_2) = (Se_1, Se_2) + \lambda(Te_1, Te_2) = \Gamma(S) + \lambda\Gamma(T)$$

so Γ is linear. For any $(v, w) \in V \times V$, there exists a *unique* map $T : \mathbf{F}^2 \to V$ such that $Te_1 = v$ and $Te_2 = w$, since e_1, e_2 is a basis of $\mathbf{F}^2$. In other words, there exists a *unique* T such that $\Gamma(T) = (v, w)$, so this shows that Γ is bijective. Thus Γ is an isomorphism.

(2) Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbf{F}))$ is injective and $\deg Tp \leq \deg p$ for all $p \in \mathcal{P}(\mathbf{F})$. Prove that T is invertible.

We need to show that T is surjective — ie, for any $q \in \mathcal{P}(\mathbf{F})$ we want to find some $p \in \mathcal{P}(\mathbf{F})$ such that $Tp = q$. Let m be some nonnegative integer such that $q \in \mathcal{P}_m(\mathbf{F})$. Observe that, since $\deg Tp \leq \deg p$ for all $p \in \mathcal{P}(\mathbf{F})$, $\mathcal{P}_m(\mathbf{F})$ is invariant under T. Since T is injective, $T|_{\mathcal{P}_m(\mathbf{F})}$ is also injective, so $T|_{\mathcal{P}_m(\mathbf{F})}$ is surjective since $\mathcal{P}_m(\mathbf{F})$ is finite dimensional. Thus there exists some $p \in \mathcal{P}_m(\mathbf{F})$ such that

$$q = (T|_{\mathcal{P}_m(\mathbf{F})})(p) = Tp.$$

This proves that T is surjective.