Problem Set 6

Note. You must provide a proof for all assertions you make in your solutions, whether the problem explicitly asks for it or not.

Problem 1. (1 point) Let X and Y be metric spaces and let E be a dense subset of X. Show that if f and g are both continuous functions $X \to Y$ such that $f(x) = g(x)$ for all $x \in E$, then in fact $f(x) = g(x)$ for all $x \in X$.

Proof. For any $a \in X$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E converging to a. Then

$$f(a) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(a). \qquad \square$$

Problem 2. (1 point) Let X, X', Y and Y' be metric spaces, and regard $X \times X'$ and $Y \times Y'$ as metric spaces using the product metric defined in problem 10 of problem set 3. Suppose $f : X \to Y$ and $f' : X' \to Y'$ are continuous functions. Show that the function $f \times f' : X \times X' \to Y \times Y'$ defined by

$$(f \times f')(x, x') = (f(x), f'(x'))$$

is also continuous.

Proof. Since every open set is a union of open balls, it suffices to show that the preimage of every open ball of $Y \times Y'$ is open. Consider an open ball

$$B_{Y\times Y'}((y, y'), r) = B_Y(y, r) \times B_{Y'}(y', r).$$

Then

$$\begin{aligned}(f \times f')^{-1}(B_{Y\times Y'}((y, y'), r)) &= \{(x, x') : f(x) \in B_Y(y, r) \text{ and } f'(x') \in B_{Y'}(y', r)\}\\ &= f^{-1}(B_Y(y, r)) \times f'^{-1}(B_{Y'}(y', r)).\end{aligned}$$

Now for any point (x, x') in this preimage, we know that $f^{-1}(B_Y(y, r))$ is an open neighborhood of x since f is continuous, so there is an open ball $B_X(x, s) \subseteq f^{-1}(B_Y(y, r))$. Similarly there is an open ball $B_X(x', s') \subseteq f'^{-1}(B_{Y'}(y', r))$. Then clearly

$$B_{X\times X'}((x, x'), \min\{s, s'\}) = B_X(x, \min\{s, s'\}) \times B'_X(x', \min\{s, s'\}) \subseteq f^{-1}(B_Y(y, r)) \times f'^{-1}(B_{Y'}(y', r)).$$

Thus (x, x') is an interior point of the preimage, so the preimage is open as well. $\square$

Problem 3. (1 point) Let X be a metric space. Show that the metric $d_X : X \times X \to \mathbb{R}$ is *uniformly* continuous, when $X \times X$ is given the product metric defined in problem 10 of problem set 2 and $\mathbb{R}$ is given the euclidean metric.

Proof. Fix $\varepsilon \gneq 0$. For any point (x, y) and (x', y') such that

$$d_{X\times X}((x, y), (x', y')) \lneq \varepsilon/2$$

observe that we have

$$d_X(x, y) \leq d_X(x, x') + d_X(x', y') + d_X(y', y) \lneq d_X(x', y') + \varepsilon$$

and similarly that

$$d_X(x', y') \lneq d_X(x, y) + \varepsilon.$$

In other words, we have

$$d_X(x, y) - \varepsilon \lneq d_X(x', y') \lneq d_X(x, y) + \varepsilon$$

so $|d_X(x, y) - d_X(x', y')| \lneq \varepsilon$. □

Problem 4. (1 point) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\limsup |a_n| = 0$. Then let $X := [0, 1]$ and for each $n \in \mathbb{N}$ consider the function $f_n : X \to \mathbb{R}$ defined as follows.

$$f_n(x) = (x + a_n)^2.$$

Does this sequence converge uniformly?

Proof. Note that absolute values are always nonnegative, so

$$0 \leq \liminf_{n\to\infty} |a_n| \leq \limsup_{n\to\infty} |a_n| = 0$$

so actually $\liminf |a_n| = \limsup |a_n| = 0$, so $\lim |a_n| = 0$, which means that $\lim a_n = 0$. Then clearly $\lim f_n(x) = x^2$ so the sequence is converging pointwise to the function f given by $f(x) = x^2$. Then

$$|f_n(x) - f(x)| = \left|2a_n x + a_n^2\right| \leq 2\,|a_n x| + \left|a_n^2\right| \leq 2\,|a_n| + |a_n|^2 .$$

In other words, we have

$$\|f_n - f\|_{\text{sup}} \leq 2\,|a_n| + |a_n|^2 .$$

But $\lim a_n = 0$ implies that $\lim(2\,|a_n| + |a_n|^2) = 0$, so for any $\varepsilon \gneq 0$, there exists N such that $2\,|a_n| + |a_n|^2 \lneq \varepsilon$ for all $n \geq N$, and then the above inequality shows that we also have $\|f_n - f\|_{\text{sup}} \lneq \varepsilon$ for all $n \geq N$. In other words, the convergence is uniform. □

Problem 5. (1 point) Let X be a metric space. Given a pair of points $x, y \in X$, a *path from x to y* is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. Then X is *path-connected* if, for every pair of points $x, y \in X$, there exists a path from x to y.

Show that, if X is path-connected, then it is also connected.

Proof. Suppose X is not connected and let U be a nonempty proper open and closed subset. Pick $x \in U$ and $y \notin U$. Then there exists a path $f : [0,1] \to X$ from x to y. Then $f^{-1}(U)$ is an open and closed subset of $[0,1]$ using continuity of f. The fact that $f(x) = 0$ means that $0 \in f^{-1}(U)$, so $f^{-1}(U)$ is nonempty. Moreover, we know that $1 \notin f^{-1}(U)$ since $f(1) = y \notin U$. Thus $f^{-1}(U)$ is a nonempty proper open and closed subset of $[0,1]$. This contradicts the fact that $[0,1]$ is connected. □

Problem 6. (3 points) Let X be an open subset of $\mathbb{R}^2$. Show that X is connected if and only if it is path-connected. *Hint.* When X is nonempty, fix a point $a \in X$ and let U be the set of $x \in X$ such that there exists a path from a to x. Show that U is open and closed in X.

Remark. It is not true for general subsets of $\mathbb{R}^2$ that connectedness implies path-connectedness. See `http://math.stanford.edu/~conrad/diffgeomPage/handouts/sinecurve.pdf` for a description of a counterexample.

Problem 7. (3 points) Let X be a metric space and suppose $(f_n)_{n\in\mathbb{N}}$ is a uniformly convergent sequence of uniformly continuous functions on X. Show that $f := \lim f_n$ is also uniformly continuous.

Problem 8. (3 points) Let X be a metric space and E a dense subset. Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions on X which converges uniformly on E. Then $(f_n)_{n\in\mathbb{N}}$ also converges uniformly on X. *Hint.* Show that $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy on X.

Proof sketch. For $x \in X$ and $t \in E$,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(t)| + |f_m(x) - f_n(t)| + |f_n(t) - f_m(t)| . \qquad \square$$

Problem 9. Let

$$X := \mathbb{R} \smallsetminus \left(\left\{-\frac{1}{n^2} : n = 1, 2, \dots\right\}\right)$$

and for each positive integer n, define the function $f_n : X \to \mathbb{R}$ as follows.

$$f_n(x) = \frac{1}{1 + n^2 x}$$

You should be able to verify that the series $\sum f_n(x)$ converges for all $x \in X \smallsetminus \{0\}$.

(a) (3 points) Describe all subsets $S \subseteq X$ such that the series of functions $\sum f_n$ is uniformly convergent on S. *Hint.* Use problem 8 to rule out some possibilities.

(b) (1 point) Let $f : X \smallsetminus \{0\} \to \mathbb{R}$ be the pointwise limit of the series of functions $\sum f_n$. Show that f is continuous.