Problem Set 5

Note. You must provide a proof for all assertions you make in your solutions, whether the problem explicitly asks for it or not.

Problem 1. (1 point) Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Hint. Notice that the nth term of the series can be rewritten as $1/n - 1/(n+1)$.

Problem 2. (1 point) Determine whether or not each of the following series converges.

$$\sum_{n=2}^{\infty} \frac{1}{(n+(-1)^n)^2}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{n!}$$

Problem 3. (1 point)

(a) Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

(b) Does there exist an *absolutely* convergent series $\sum a_n$ for which $\sum a_n^2$ diverges?

Problem 4. (1 point) Determine the set of all $x \in \mathbb{R}$ for which the following power series converges.

$$\sum_{n=0}^{\infty} \left(\frac{4+2(-1)^n}{5} \right)^n x^n$$

Problem 5. (1 point) Let X be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if the series

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1})$$

converges.

Problem 6. (1 point) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\liminf |a_n| = 0$. Show that there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that the series

$$\sum_{k=0}^{\infty} a_{n_k}$$

converges absolutely.

Problem 7. (3 points) Show that if $(a_n)_{n\in\mathbb{N}}$ is a monontonically decreasing sequence in $\mathbb{R}$ and $\sum a_n$ converges, then $\lim na_n = 0$. *Remark.* Notice that this result gives another proof that the harmonic series diverges.

Problem 8. (3 points) Is it true that, for any closed subset $E \subseteq \mathbb{R}$, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ whose set of subsequential limits in $\mathbb{R}$ is exactly equal to E?

Problem 9. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection and suppose further that there exists a strictly increasing sequence $N_0 \lneq N_1 \lneq \cdots$ of natural numbers such that $\sigma(\{0, \ldots, N_i\}) \subseteq \{0, \ldots N_i\}$ for all i. Further, let $\sum a_n$ be a convergent series.

(a) (2 points) Show that, if the rearrangement $\sum a_{\sigma(n)}$ also converges, then

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_{\sigma(n)}.$$

(b) (2 points) Give an example to show that the rearrangement $\sum a_{\sigma(n)}$ need not converge.

(c) (2 points) Show that, if there exists a constant M such that $N_{i+1} - N_i \leq M$ for all i, then the rearrangement $\sum a_{\sigma(n)}$ must converge.