Problem Set 4

Note. You must prove all assertions you make, whether the problem explicitly asks for it or not.

Problem 1. (1 point) Suppose $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences in $\mathbb{R}$ and $|x_n| \leq y_n$ for all n. If $\lim y_n = 0$, show that $\lim x_n = 0$ also.

Problem 2. (1 point) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space X and, for any $\varepsilon \gneq 0$ and $a \in X$, define

$$J_\varepsilon(a) := \{n \in \mathbb{N} : x_n \in B(a,\varepsilon), x_n \neq a\}.$$

(a) Show that, if $J_\varepsilon(a)$ is nonempty for all $\varepsilon \gneq 0$, then a is a subsequential limit of $(x_n)_{n\in\mathbb{N}}$.

(b) Show that statement of part (a) would be false if we had omitted the condition "$x_n \neq a$" from the definition of $J_\varepsilon(a)$.

Problem 3. (1 point) Show that a closed subset F of a complete metric space X is also complete.

Problem 4. (1 point) Let X be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X.

(a) Show that $(x_n)_{n\in\mathbb{N}}$ converges to a point $x \in X$ if and only if every subsequence of $(x_n)_{n\in\mathbb{N}}$ has further subsequence which converges to x.

(b) If $X = \mathbb{R}$, show that $\lim x_n = \infty$ if and only if every subsequence of $(x_n)_{n\in\mathbb{N}}$ has a further subsequence whose limit is ∞.

(c) Give an example of a sequence $(x_n)_{n\in\mathbb{N}}$ which does not converge, but which has the property that every subsequence has a further convergent subsequence.

Problem 5. (1 point) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let

$$E := \left\{a \in \mathbb{R} \cup \{\infty, -\infty\} : \lim_{k\to\infty} x_{n_k} = a \text{ for some subsequence } (x_{n_k})_{k\in\mathbb{N}}\right\}.$$

Show that E is a singleton set $\{a\}$ if and only if $\lim x_n = a$. *Hint.* You might use problem 4.

Problem 6. (1 point) Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be bounded sequences in $\mathbb{R}$.

(a) Show that

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

(b) Give an example to show that the inequality in part (a) can be strict.

Problem 7. (1 point for each part) This problem describes an algorithm for approximating square roots and investigates its rate of convergence. For some positive real number a, suppose that $x_0 \gneq \sqrt{a}$ and then define

$$x_{n+1} = \frac{x_n^2 + a}{2x_n}.$$

(a) Prove that $(x_n)_{n\in\mathbb{N}}$ is monotonically decreasing and that $\lim x_n = \sqrt{a}$.

(b) Let $\varepsilon_n := x_n - \sqrt{a}$. Show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n}.$$

(c) Describe happens if I accidentally choose x_0 so that $0 \lneq x_0 \lneq \sqrt{a}$. Does the sequence still converge?

Takeaway. Notice that $x_n \gneq \sqrt{a}$, so if we let $\beta := 2\sqrt{a}$, and induct on part (b), we find that $\varepsilon_n \lneq \beta(\varepsilon_0/\beta)^{2^n}$. In other words, if we start with a reasonable estimate x_0 for $\sqrt{a}$ (more precisely, one for which the initial error ε_0 is less than the constant β, so that $\varepsilon_0/\beta \lneq 1$), the terms x_n of this sequence get close to $\sqrt{a}$ very very quickly! (In fact, even if if my initial estimate is bad and not within β of $\sqrt{a}$, the fact that $\lim x_n = \sqrt{a}$ tells us that *eventually* I'll wind up with some x_n such that $\varepsilon_n \lneq \beta$, and then after that I'll still have rapid convergence.)

Problem 8. Let X be a complete metric space.

(a) (1 point) Let $E_0 \supseteq E_1 \supseteq \cdots$ be a nested infinite chain of closed and bounded subsets of X such that $\lim \operatorname{diam}(E_n) = 0$. Show that $\bigcap_{n\in\mathbb{N}} E_n$ contains just a single point. *Hint.* Recall the proof we gave in class of the fact that complete and totally bounded implies compact.

(b) (2 points) Show that if $G_0, G_1, \ldots$ is a countable collection of dense open subsets of X, then $\bigcap_{n\in\mathbb{N}} G_n$ is also dense in X. *Hint.* Use part (a).

Problem 9. (3 points) Let S be a set and let X be the set of bounded functions $S \to \mathbb{R}$ regarded as a metric space with the supremum metric. Show that X is complete. (Thus, using problem 3, we can deduce that the closed subset $F := \{f \in X : \|f\|_{\sup} \leq 1\}$ is complete. Recall that we showed on problem set 3 that, when S is infinite, then F is *not* compact. It follows that F must not be totally bounded.)

Problem 10. (5 points) Let S be a set. Let us say that a metric d on S is *bounded* if there exists a real number R such that $d(x, y) \leq R$ for all $x, y \in S$. Let X be the set of all bounded functions $S \times S \to \mathbb{R}$ regarded as a metric space with the supremum metric. Notice that the set M of bounded metrics on S is a subset of the metric space X. Is M closed in X? If so, prove it. If not, describe all of the elements of the closure $\bar{M}$.

Further problems. If you're familiar with the concept of equivalence relations and of quotients of sets by equivalence relations, I highly recommend that you work through exercises 23, 24, and 25 in chapter 3 of Rudin at some point when you have time.