Problem Set 3

Problem 1. Let X be a metric space. For any $a \in X$ and positive real number r, let

$$B^+(a,r) := \{x \in X : d(a,x) \leq r\}.$$

(a) Show that $B^+(a,r)$ is bounded.

(b) Show that $B^+(a,r)$ is closed in X.

(c) Must $B^+(a,r)$ be equal to the closure of $B(a,r)$?

Proof. For any $x, y \in B^+(a,r)$, we have

$$d(x,y) \leq d(x,a) + d(a,y) \leq r + r \leq 2r$$

so $\operatorname{diam} B^+(a,r) \leq 2r$. Thus $B^+(a,r)$ is bounded.

Suppose $x \notin B^+(a,r)$. Then $d(a,x) \gneq r$, so choose some ε such that $0 \lneq \varepsilon \leq d(a,x) - r$. Then for any $y \in B(x,\varepsilon)$, if $y \in B^+(a,r)$ then we find that

$$d(a,x) \leq d(a,y) + d(y,x) \lneq r + \varepsilon \leq r + d(a,x) - r = d(a,x)$$

which is a contradiction. Thus $B(x,\varepsilon) \subseteq X \setminus B^+(a,r)$, proving that every point of the complement $X \setminus B^+(a,r)$ is an interior point, so the complement is open, so $B^+(a,r)$ is closed.

Finally, $B^+(a,r)$ need not be equal to the closure of $B(a,r)$. Consider X being some set with at least 2 elements and the discrete metric. Then $B(a,1) = \{a\}$ is closed so is equal to its own closure, but $B^+(a,1) = X \neq B(a,1)$. $\square$

Problem 2. (1 point) Let X be a metric space, E a subset of X, and S a subset of E. Consider the following statements.

(a) If E is open in X, and S is open in E, then S is open in X.

(b) If E is open in X, and S is closed in E, then S is closed in X.

(c) If E is closed in X, and S is open in E, then S is open in X.

(d) If E is closed in X, and S is closed in E, then S is closed in X.

Determine whether each of the above statements is true or false. If it is true, prove it. If it is false, provide a counterexample. *Remark.* There are another four statements of this form that one could make. You might optionally try listing those four statements as well and thinking about whether those are true or not.

Proof. (a) True. Since S is open in E, there exists an open subset U of X such that $S = E \cap U$. But E and U are both open in X, so the intersection S is also open in S.

(b) False. For instance, $E := (0, 1)$ is open in $X := \mathbb{R}$ with the euclidean metric, and $S := E$ is closed in E, but S is not closed in X.

(c) False. For instance, $E := \{0\}$ is closed in $X := \mathbb{R}$ with the euclidean metric, and $S := E$ is open in E, but S is not open in X.

(d) True. Since S is closed in E, we know that $E \smallsetminus S$ is open in E, which means that there exist some open U inside X such that $E \smallsetminus S = U \cap E$. Now notice that

$$X \smallsetminus S = (E \smallsetminus S) \cup (X \smallsetminus E) = (U \cap E) \cup (X \smallsetminus E) = U \cup (X \smallsetminus E)$$

and U and $X \smallsetminus E$ are both open in X, so their union is also open in X. Thus we conclude that S is closed in X.

$\square$

Problem 3. (1 point) Let X be a metric space and suppose that $K_1, \dots, K_n$ are compact subsets of X. Show that

$$K := K_1 \cup \cdots \cup K_n$$

is also compact.

Proof. Let $\mathfrak{U}$ be a collection of open subsets of X whose union contains K. Then the union of the open sets in $\mathfrak{U}$ also contains K_i for each i, so compactness of K_i means that there exists a finite subcollection $\mathfrak{U}_i$ whose union still contains K_i. Then $\mathfrak{U}' := \mathfrak{U}_1 \cup \cdots \cup \mathfrak{U}_n$ is a finite subcollection of $\mathfrak{U}$ whose union contains K. Thus K is compact. $\square$

Problem 4. (1 point) Let X be a compact metric space. Show that X is bounded.

Proof. Consider the collection $\mathfrak{U} := \{B(x, 1) : x \in X\}$. This is an open cover of X, so it has a finite subcover. In other words, there exist finitely many points $x_1, \dots, x_n$ such that

$$X = B(x_1, 1) \cup \cdots \cup B(x_n, 1).$$

Let $R := \max\{d(x_i, x_j) : i, j = 1, \dots, n\}$. Now for any $x, y \in X$, notice that we have $x \in B(x_i, 1)$ for some i and $y \in B(x_j, 1)$ for some j, so then

$$d(x, y) \leq d(x, x_i) + d(x_i, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) \lneq 1 + R + 1 = R + 2.$$

Thus X is bounded. $\square$

Problem 5 (1 point)**.** Let X be a metric space and suppose a set I indexes a collection $(K_i)_{i\in I}$ of compact subsets $K_i \subseteq X$. Show that $K := \bigcap_{i\in I} K_i$ is also compact.

Proof. Each K_i is closed in X, so K is also closed in X since intersections of closed sets are closed. Since K is closed in X, we know that $X \setminus K$ is open, so if we fix some $i \in I$, then $(X \setminus K) \cap K_i$ is open in K_i, which means that

$$K_i \setminus ((X \setminus K) \cap K_i) = K$$

is closed in K_i. Thus K is a closed subset of the compact space K_i, so it is compact. $\square$

Problem 6 (1 point)**.** Let E be a subset of the closed interval $[0, 1]$ and let E' denote its set of limit points. We showed in class that when E is the Cantor set, E is closed, that $E \subseteq E'$ and that $E^\circ = \emptyset$. The point of this problem is for you to come up with other examples of sets which satisfy any two of these properties but not all three of them.

In other words, give an example of a subset E inside $[0, 1]$ such that...

(a) E is closed and $E \subseteq E'$, but $E^\circ \neq \emptyset$.

(b) E is closed and $E^\circ = \emptyset$, but $E \not\subseteq E'$.

(c) $E \subseteq E'$ and $E^\circ = \emptyset$, but E is not closed.

Proof. For (a), $E = [1/4, 3/4]$ or any other closed interval would work. For (b), $E = \{0\}$ works, and for (c), $E = \mathbb{Q} \cap [0, 1]$ works. $\square$

Problem 7 (1 point)**.** A metric space X is *totally disconnected* if the only connected subsets of X are the empty set and singleton sets.

Show that the Cantor set is totally disconnected. *Hint.* You can do this using ternary expansions if you want, but there is a much simpler proof that doesn't use ternary expansions: recall that we know precisely when subsets of $\mathbb{R}$ are connected.

Proof. It is easy to see that the empty set and the singleton sets are connected subsets. Conversely, let E be a subset of the Cantor set X containing two distinct points x and y, and assume without loss of generality that $x \lneq y$. Consider the open set (x, y). If this set was contained in X, then any $z \in (x, y)$ would be an interior point of X. But we proved in class that X has empty interior inside $\mathbb{R}$, so $(x, y) \not\subseteq X$. Thus there exists some $z \in (x, y) \setminus X$. In particular, $z \notin E$ also. Thus, we have $x, y \in E$ but $[x, y] \not\subseteq E$ since $z \in [x, y] \setminus E$, so, using the characterization of connected subsets of $\mathbb{R}$ we proved in class, we conclude that E is disconnected. $\square$