Problem Set 1

Problem 1. (1 point) Prove that there exists no rational number whose square is 75.

Problem 2. (1 point) Let E be a nonempty subset of an ordered set X. Suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Problem 3. (1 point) Let X be an ordered set with the supremum property, and suppose S and T are nonempty bounded subsets of X such that T is bounded and $S \subseteq T$. Show that S is bounded, and that

$$\inf T \leq \inf S \leq \sup S \leq \sup T.$$

Problem 4. (1 point) Are there any nonempty subsets S of $\mathbb{R}$ such that $\inf S = \sup S$? If so, describe all of them and prove that you have described all of them. If not, prove that there are none.

Problem 5. (1 point) Let A be a nonempty subset of $\mathbb{R}$ and let

$$-A := \{-x : x \in A\}.$$

Prove that $\inf(A) = -\sup(-A)$. *Hint.* There are no boundedness assumptions on A in this statement. So, first consider the case when A is not bounded below, in which case $\inf(A) = -\infty$, and then consider the case when A is bounded below.

Problem 6. (3 points) Determine whether or not $\sqrt{4 + 2\sqrt{3}} - \sqrt{3}$ is rational.

Problem 7. (5 points) Fix a real number $b \gneq 1$. The point of this problem is to give a rigorousous definition of b^x when x is an arbitrary real number (there are other rigorous definitions as well). Let us recap what we do already know how to define. We know that $b^0 = 1$. Now let n be a positive integer. We know that b^n is defined to be the product of b with itself n times. We also know that b^{-n} is defined to be $1/b^n$. Finally, we know that $b^{1/n}$ is defined to be the unique real number α such that $\alpha^n = b$ (we proved in class that such a real number α exists and that it is unique).

(a) Let r be a rational number and suppose that p, q, p', q' are integers such that $q, q' \gneq 0$ and $r = p/q = p'/q'$. Prove that

$$(b^p)^{1/q} = (b^{p'})^{1/q'}.$$

Hence it makes sense to define $b^r := (b^p)^{1/q}$.

(b) Prove that $b^{r+s} = b^r b^s$ when r and s are rational.

(c) For any real number x, define the set

$$B(x) := \{b^t : t \in \mathbb{Q} \text{ and } t \leq x\}.$$

Prove that $b^r = \sup B(r)$ for any rational number r.

Hence, it makes sense to define $b^x := \sup B(x)$ for every real number x.

(d) Prove that $b^{x+y} = b^x b^y$ for all real numbers x and y.

Problem 8. (5 points) Let $\mathbb{Q}(\sqrt{2}) := \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

(a) Verify that the set $\mathbb{Q}(\sqrt{2})$ is a subfield of $\mathbb{R}$.

(b) Is the usual order on $\mathbb{Q}(\sqrt{2})$ the only order which makes $\mathbb{Q}(\sqrt{2})$ an ordered field? If so, prove it. If not, describe all other orders and prove that you have described all of them.