Sequences

1 Sequences

Let X be a metric space. A *sequence* in X is collection $(x_n)_{n\in\mathbb{N}} = (x_0, x_1, x_2, \dots)$ where $x_n \in X$ for all $n \in \mathbb{N}$. A point $x \in X$ is a *limit* of the sequence $(x_n)_{n\in\mathbb{N}}$ if, for every open set U containing x, there exists some natural number N such that $x_n \in U$ for all $n \geq N$. If x is a limit of $(x_n)_{n\in\mathbb{N}}$, we say that $(x_n)_{n\in\mathbb{N}}$ *converges to* x. If $(x_n)_{n\in\mathbb{N}}$ has a limit, we say that it is *convergent*.

Example 1.1. Let $X := \mathbb{R}$ and $x_n = 1/(n+1)$ for all n, so that we have the sequence

$$(1, 1/2, 1/3, 1/4, \dots)$$

and 0 is a limit of this sequence. Indeed, given any open set U containing 0, there exists an open ball $B(0,r) \subseteq U$. By the archimedean property there exists some N such that $1/N \lneq r$. But then

$$x_n = 1/(n+1) \lneq 1/n \leq 1/N \lneq r$$

for all $n \geq N$, which means that $x_n \in B(0,r) \subseteq U$ for all $n \geq N$.

Example 1.2. Let $X := \mathbb{R}$ and $x_n = (-1)^n$ for all $n \in \mathbb{N}$. In other words, we consider the sequence

$$(1, -1, 1, -1, \dots).$$

This sequence is not convergent. Indeed, consider first any $a \geq 0$. Then $-1 \notin B(a,1)$, so there does *not* exist some N sufficiently large that $x_n \in B(a,1)$ for all $n \geq N$. On the other hand, for $a \lneq 0$, we have $1 \notin B(a,1)$, so again there does *not* exist some N such that $x_n \in B(a,1)$ for all $n \geq N$. Thus no $a \in \mathbb{R}$ is a limit of this sequencce, so this sequence is not convergent.

Lemma 1.3 (Uniqueness of limits)**.** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space X. If x and x' are both limits of $(x_n)_{n\in\mathbb{N}}$, then $x = x'$. Thus, it makes sense to define the expression*

$$\lim_{n\to\infty} x_n$$

to refer to the unique limit of a convergent sequence $(x_n)_{n\in\mathbb{N}}$.

Proof. Pick some positive real number ε. Then $B(x, \varepsilon/2)$ is an open set containing x, so there exists some N such that $x_n \in B(x, \varepsilon/2)$ for all $n \geq N$. Similarly there exists some N' such that $B(x', \varepsilon/2)$

contains x_n for all $n \geq N'$. Then for all $n \geq \max\{N, N'\}$, we have $x_n \in B(x, \varepsilon/2) \cap B(x', \varepsilon/2)$. This means that

$$d(x, x') \leq d(x, x_n) + d(x_n, x') \lneq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Notice that this is true for all positive real numbers ε. In other words, $d(x, x')$ is a lower bound for the set $(0, \infty)$, which means that $d(x, x') \leq 0$ since $0 = \inf(0, \infty)$. But $d(x, x') \geq 0$ by the positivity axiom (M1) for metrics, so $d(x, x') = 0$, and this in turn implies that $x = x'$ by axiom (M4) for metrics. $\square$

Remark 1.4. This last paragraph of the above proof is far more detailed than you'll usually see in the mathematical literature. Usually, this kind of logic would be abbreviated to something like "Since $\varepsilon \gneq 0$ was arbitrary, $d(x, x') = 0$." This comes up very often. See problem 1 for another example.

Remark 1.5. Limit points and limits are closely related, but, despite the similarity of the terminology, they are not identical concepts. You should work through problems 2, 3 and 4 to get your bearings straight.

2 Subsequences

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space X and let $n_1 \lneq n_2 \lneq n_3 \lneq \cdots$ be an infinite increasing sequence of integers. A *subsequence* of $(x_n)_{n\in\mathbb{N}}$ is a sequence of the form $(x_{n_k})_{k\in\mathbb{N}}$.

Example 2.1. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ where $x_n = (-1)^n$ for all $n \in \mathbb{N}$. Then the infinite increasing sequence of integers $n_k = 2k$ gives rise to the subsequence

$$(x_0, x_2, x_4, \dots) = (1, 1, 1, \dots)$$

and this new sequence $(x_{n_k})_{k\in\mathbb{N}}$ evidently converges to 1, even though we already saw before that $(x_n)_{n\in\mathbb{N}}$ does not itself converge.

If $(x_n)_{n\in\mathbb{N}}$ is a sequence, then the limit of some subsequence of $(x_n)_{n\in\mathbb{N}}$ is called a *subsequential limit* of $(x_n)_{n\in\mathbb{N}}$.

We can reformulate this definition as follows. Suppose we've fixed a sequence $(x_n)_{n\in\mathbb{N}}$ as well as a point $a \in X$. I challenge you by telling you a distance from a as well as a position in the sequence, and asking that you find some point of the sequence which is further along than the position I told you *and* closer to a than the distance I told you. If you can always do this, no matter what distance and position I tell you, then, and only then, is a a subsequential limit. This is made precise by lemma 2.2.

Lemma 2.2. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space X and let $a \in X$ be some point. Then a is a subsequential limit of $(x_n)_{n\in\mathbb{N}}$ if and only if, for every open set U containing a and every $N \in \mathbb{N}$, there exists some $n \geq N$ such that $x_n \in U$.*

Proof. Suppose a is a subsequential limit, so there exists some natural numbers $n_0 \lneq n_1 \lneq \cdots$ such that $(x_{n_k})_{k\in\mathbb{N}}$ converges to a. Let U be an open set containing a and $N \in \mathbb{N}$. Since $(x_{n_k})_{k\in\mathbb{N}}$

converges to a, by definition of convergence there exists some K such that, for all $k \geq K$, we have $x_{n_k} \in U$. Now since $n_0 \lneq n_1 \lneq \cdots$, there must also exist some $k \geq K$ such that $n_k \geq N$, and we have found the desired point of the sequence.

For the converse, we must construct a subsequence converging to a. To start things off, let $n_0 := 0$. Inductively, suppose we have chosen $n_0, \ldots, n_{k-1}$, and consider the open ball $B(a, 1/k)$. By our assumption on a, there exists some $n_k \geq n_{k-1} - 1$ such that $x_{n_k} \in B(a, 1/k)$. Now consider the resulting sequence $(x_{n_k})_{k\in\mathbb{N}}$. We claim that it converges to a. Indeed, suppose U is any open set containing a. Then there exists some $\varepsilon \gneq 0$ such that $B(a, \varepsilon) \subseteq U$. But there exists some K such that $1/K \lneq \varepsilon$. Then for all $k \geq K$, we have $1/k \leq 1/K \lneq \varepsilon$, which means that

$$x_{n_k} \in B(a, 1/k) \subseteq B(a, \varepsilon) \subseteq U.$$

Thus $(x_{n_k})_{k\in\mathbb{N}}$ converges to a. □

Corollary 2.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space in X and let E be the set of all of its subsequential limits. Then E is closed.*

Proof. We'll show that $X \setminus E$ is open, so suppose $a \in X \setminus E$. In other words, a is *not* a subsequential limit. By lemma 2.2, this means that there exists some open set U containing a and some position $N \in \mathbb{N}$ such that $x_n \notin U$ for any $n \geq N$. But then, by the same lemma, no other point of U can be a subsequential limit either! In other words, $U \subseteq X \setminus E$. But then we can find an open ball $B(a, r) \subseteq U \subseteq X \setminus E$, and this shows that a is an interior point of $X \setminus E$. □

For a different, more hands-on proof of corollary 2.3, see the proof of theorem 3.7 in Rudin.

3 Cauchy Sequences

Let X be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ is *Cauchy* if, for every $\varepsilon \gneq 0$, there exists some $N \in \mathbb{N}$ such that $d(x_m, x_n) \lneq \varepsilon$ whenever $m, n \geq N$.

Lemma 3.1. *If $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in a metric space X, then $(x_n)_{n\in\mathbb{N}}$ is also Cauchy.*

Proof. Let $\varepsilon \gneq 0$ and let $a = \lim x_n$. Since the sequence is convergent, there exists N such that $x_n \in B(a, \varepsilon/2)$ for all $n \geq N$. Then notice that for all $m, n \geq N$, we have

$$d(x_m, x_n) \leq d(x_m, a) + d(a, x_n) \lneq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$ □

The converse of lemma 3.1 need not be true in general. Before we get to an example of this phenomenon, let us make a definition: a metric space X is *complete* if every Cauchy sequence is convergent. Two very important facts are that compact metric spaces are complete, and that $\mathbb{R}^n$ with the euclidean metric is complete. We will prove these facts tomorrow. For now, let us discuss a non-example.

Example 3.2. We can construct a Cauchy sequence which does not converge as follows. Let $a := \sqrt{2}$, or any other irrational number. For any $n \in \mathbb{N}$, density of the rationals guarantees that there exists a rational number x_n such that

$$a - 1/(n+1) \lneq x_n \lneq a.$$

Now note that for any open ball $B(a, \varepsilon)$, there exists some N such that $1/(N+1) \lneq \varepsilon$, and then

$$d(a, x_n) \leq 1/(n+1) \leq 1/(N+1) \lneq \varepsilon$$

for all $n \geq N$, so $x_n \in B(a, \varepsilon)$. This shows that the sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers converges to a in $\mathbb{R}$, so by lemma 3.1, it must be Cauchy.

But instead of regarding this as a sequence in $\mathbb{R}$, let us regard $(x_n)_{n\in\mathbb{N}}$ as a sequence in $\mathbb{Q}$. If this sequence had a limit in $\mathbb{Q}$, then it would also be a limit in $\mathbb{R}$, but we know that limits are unique by lemma 1.3 and we explicitly chose a to be irrational. So this sequence cannot have a limit in $\mathbb{Q}$. Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space $\mathbb{Q}$ which does not converge. We conclude that $\mathbb{Q}$ is not a complete metric space.

Remark 3.3. The above example might seem a bit contrived since we took a sequence which converges in a larger space but then just forgot about the larger space, but it's actually not contrived at all. It turns out that any metric space sits inside a bigger complete metric space called its *completion*. So any time you run into a non-convergent Cauchy sequence, its because you're looking for a limit in too small a space, and you'll find that limit in a bigger space.

In fact, it turns out that $\mathbb{R}$ is the completion of $\mathbb{Q}$. One can even define $\mathbb{R}$ in this way, but to make sense of this approach of defining $\mathbb{R}$, one has to talk about something a big more general than metric spaces. Talk to me in office hours if you're curious about this.

4 Sample Problems

Problem 1. Let X be a metric space and E a bounded subset of X. Then $\bar{E}$ is also bounded, and in fact $\operatorname{diam}(\bar{E}) = \operatorname{diam}(E)$.

Solution. Let $R := \operatorname{diam}(E)$. Since $E \subseteq \bar{E}$, it is clear that $R \leq \operatorname{diam}(\bar{E})$, so we just need to show that $d(a, b) \leq R$ for all $a, b \in \bar{E}$. Fix some $\varepsilon \gneq 0$. Then the open ball $B(a, \varepsilon/2)$ contains at least one point x that is also in E. This point could be a itself if it happens that $a \in E$, but if $a \notin E$, then a is a limit point of E so we can still find such an x. Similarly, we can also choose a point $y \in B(b, \varepsilon/2) \cap E$. Then

$$d(a, b) \leq d(a, x) + d(x, b) \leq d(a, x) + d(x, y) + d(y, b) \lneq \varepsilon/2 + R + \varepsilon/2 = R + \varepsilon.$$

Since $\varepsilon \gneq 0$ is arbitrary, we conclude that $d(a, b) \leq R$. □

Remark. Note that the above problem is not about sequences at all, but it does use this "since $\varepsilon \gneq 0$ is arbitrary" trick that we saw in the proof of lemma 1.3, which is very important to get used to. You should make sure you can turn the last sentence of the above solution into a full-fledged formal proof.

Problem 2. Let E be a subset of a metric space X.

(a) Show that if $a \in X$ is a limit point of E, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in E whose limit is a.

(b) Use part (a) to conclude that a point $a \in X$ is an element of $\bar{E}$ if and only if there exists a sequence in E whose limit is a.

Problem 3. Let X be a metric space and E a subset. Give an example of a sequence $(x_n)_{n\in\mathbb{N}}$ in E whose limit is some point $a \in X$, but a is not a limit point of E.

Hint. Consider the $X := \mathbb{R}$, $E := \{0\}$, and the sequence $(x_n)_{n\in\mathbb{N}}$ where $x_n = 0$ for all $n \in \mathbb{N}$.

Problem 4. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space X and consider the set $E := \{x_n : n \in \mathbb{N}\}$.

(a) Give an example of a situation when some point $a \in X$ is a limit point of E even though a is not the limit of the sequence $(x_n)_{n\in\mathbb{N}}$.

(b) Give an example of a situation when a is the limit of the sequence $(x_n)_{n\in\mathbb{N}}$ even though a is not a limit point of E.

Hint. For (a), consider the sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ where $x_{2k} = 1/(k+1)$ and $x_{2k+1} = 1$ for all k. For (b), consider the sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ where $x_n = 0$ for all $n \in \mathbb{N}$.

Problem 5. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in a metric space X. Show that the set $E := \{x_n : n \in \mathbb{N}\}$ is bounded.

Hint. Let a be the limit of the sequence and consider $U := B(a,1)$. Then U contains almost all of the points of E. Choose some real number R that is bigger than 1 and is also bigger than $d(a, x_n)$ for the finitely many x_n such that $x_n \notin U$. Now find an upper bound for $d(x_m, x_n)$ using the triangle inequality. This upper bound should be some expression involving R.

Problem 6. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in a metric space X and suppose a is a subsequential limit of $(x_n)_{n\in\mathbb{N}}$. Then $(x_n)_{n\in\mathbb{N}}$ converges to a.

Proof. Let U be some open set containing a, and let $\varepsilon \gneq 0$ be such that $B(a,\varepsilon) \subseteq U$. Since the sequence is Cauchy, there exists an N such that $d(x_m, x_n) \lneq \varepsilon/2$ for all $m, n \geq N$. Since a is a subsequential limit, there exists some $m \geq N$ such that $x_m \in B(a, \varepsilon/2)$. Then for all $n \geq N$, we have

$$d(a, x_n) \leq d(a, x_m) + d(x_m, x_n) \lneq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

so $x_n \in B(a,\varepsilon) \subseteq U$. □

Problem 7. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in a metric space X. Show that the set $E := \{x_n : n \in \mathbb{N}\}$ is bounded.

Hint. There exists an N such that $d(x_m, x_n) \lneq 1$ for all $m, n \geq N$. Now note that the set $\{d(x_m, x_n) : m, n \leq N\}$ is finite, so let R be some number that is both larger than every element of this set and is larger than 1. Then explain why $\operatorname{diam}(E) \leq R$.