

Cantor Set

The Cantor set is an specific subset of \mathbb{R} which has lots of weird and surprising properties.

1 Construction

Let $E_0 := [0, 1]$. We then remove the open middle third from the interval E_0 to get the set E_1 . In other words,

$$E_1 := [0, 1/3] \cup [2/3, 1].$$

Now to get E_2 , we remove the open middle third from each of the two segments $[0, 1/3]$ and $[1/3, 1]$. In other words,

$$E_2 := [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

We keep going like this. Inductively, we construct a sequence of sets

$$E_0 \supseteq E_1 \supseteq E_2 \supseteq \dots$$

such that E_n is the union of 2^n disjoint closed intervals, each of diameter 3^{-n} . The *Cantor set* is the set

$$E := \bigcap_{n \in \mathbb{N}} E_n,$$

and we regard it as a metric space by restricting the euclidean metric on \mathbb{R} .

2 Alternative Construction

We can express this construction slightly differently using ternary expansions. There will be many unproved assertions in this section. For this reason, you should note that everything that we can prove about the Cantor set using ternary expansions can also be proved without ternary expansions. But sometimes it's a bit easier to think about this alternative construction using ternary expansions, so it's useful to know this alternative construction even though we won't prove everything (or really anything at all) in this section formally.

We haven't proved this in class, but hopefully you know that any number in $[0, 1]$ has a *ternary expansion* of the form

$$a.a_0a_1a_2a_2\dots$$

where a is either 0 or 1, and $a_i \in \{0, 1, 2\}$ for all $i \in \mathbb{N}$. If you want to see a proof of this fact, note that in section 1.22, Rudin explains why decimal expansions exist. The proof that ternary

expansions exist is identical, but with the number 3 in place of the number 10 everywhere.

A fact that you may be familiar with is that some numbers have multiple ternary expansions. For example, the usual ternary expansion of the number 1 is $1.0000\dots$, but $0.2222\dots$ is an alternative ternary expansion. (The analogous fact for decimal expansions, instead of ternary expansions, is $0.9999\dots = 1$.) The fact that these two ternary expansions represent the same number is something we will prove later on in this class. For now, you should just accept that these two ternary expansions represent the same number. Here are two other examples of non-unique ternary expansions.

$$1/3 = 0.10000\dots = 0.022222\dots$$

$$7/9 = 0.21000\dots = 0.202222\dots$$

Now notice that E_1 is precisely the set of real numbers which have a modified ternary expansion that does not have a 1 in the first place after the decimal point. Indeed, the numbers in $[0, 1/3]$ can all be written in the form $0.0a_1a_2a_3\dots$, and the numbers in $[2/3, 1]$ can all be written in the form $0.2a_1a_2a_3\dots$. Then E_2 is the set of all real numbers which have a ternary expansion that does not have a 1 in either of the first 2 decimal places. For example, the numbers in $[0, 1/9]$ can all be written in the form $0.00a_2a_3\dots$, and the numbers in $[2/9, 1/3]$ can all be written in the form $0.02a_2a_3\dots$, and so forth. In general, the numbers in E_n all have ternary expansions which do not have any 1's anywhere in the first n places after the decimal. In other words,

$$E_n := \{0.a_0a_1\dots : a_0, a_1, \dots \in \{0, 1, 2\}, a_0, \dots, a_{n-1} \neq 1\}.$$

Thus the Cantor set E can also be described as the set of all numbers which have a ternary expansion containing no 1's.

Notice that if a number has a ternary expansion containing no 1's, it actually has only one such ternary expansion. For example, we know that $2/3$ has a ternary expansion $0.2000\dots$ which does not contain any 1's. This number has only one other ternary expansion, which is $0.1222\dots$, and this ternary expansion does contain a 1.

3 Properties

Here are some properties of the Cantor set. Each of these properties individually is not so unusual: it's easy to think of uncountable subsets of \mathbb{R} , and compact subsets of \mathbb{R} , and "perfect" subsets of \mathbb{R} (defined below), and subsets of \mathbb{R} which have empty interior... No one of these properties is special. The Cantor set is unusual because it satisfies *all* of these properties.

- The Cantor set E is nonempty. For example, notice that $0 \in E_n$ for all n , so $0 \in E$. Alternatively, notice that whenever we remove a middle third, the endpoints of that removed bit remain fixed in all future stages. For example, in the first step, we removed $(1/3, 2/3)$ from $E_0 := [0, 1]$, and then $1/3$ and $2/3$ remain fixed in the sense that they are both elements of E_n for all $n \geq 0$, and therefore are both elements of E .

In fact, not only is E nonempty, it is even uncountable. You can prove this without appealing

to ternary expansions (if you'd like to see this, take a look at Rudin's theorem 2.43), but we'll give a more elementary proof of uncountability here using ternary expansions. The argument is basically identical to the usual Cantor's diagonalization argument for proving uncountability.

Suppose the Cantor set E were countable, so that we have a list x_0, x_1, \dots containing all of the elements of E . Each of these elements has a ternary expansion containing no 1's, so let us write out these ternary expansions.

$$\begin{aligned} x_0 &= 0.a_{0,0}a_{0,1}a_{0,2}\cdots \\ x_1 &= 0.a_{1,0}a_{1,1}a_{1,2}\cdots \\ &\vdots \\ x_i &= 0.a_{i,0}a_{i,1}a_{i,2}\cdots \\ &\vdots \end{aligned}$$

Now construct a number y as follows. We know that $a_{0,0}$ is either 0 or 2, so let b_0 be whichever of the two values (0 or 2) that $a_{0,0}$ is not. Again, we know that $a_{1,1}$ is either 0 or 2, so let b_1 be whichever of the two values that $a_{1,1}$ is not. In general, let b_n be equal to the unique element of the set $\{0, 2\} \setminus \{a_{n,n}\}$. Now consider

$$y := 0.b_0b_1b_2\cdots$$

Notice that each b_i is either 0 or 2, so y has a ternary expansion which contains no 1's, so y is an element of E . On the other hand, $y \neq x_i$ for all i . Indeed, if $y = x_i$ for some i , then $0.a_{i,0}a_{i,1}\cdots$ and $0.b_0b_1\cdots$ are two distinct ternary expansions for the same number, both of which contain no 1's, and we already noted before that this cannot happen.

- E is compact. See problem 1.
- Since E is a compact subset of \mathbb{R} , it must be closed in \mathbb{R} . In particular, it contains all of its limit points. In fact, more is true: all of its points are limit points (in other words, E is a *perfect* subset of \mathbb{R}). To see this, suppose $a \in E$ and consider an arbitrary open ball $B(a, r)$. Then for n large enough, we know that $3^{-n} \lesssim r$. Since $a \in E \subseteq E_n$, and E_n is the union of several disjoint closed intervals, one of these intervals I_n contains a . Moreover, $\text{diam}(I_n) = 3^{-n}$, so if x is one of the endpoints of I_n distinct from a , then

$$d(a, x) \leq \text{diam}(I_n) = 3^{-n} \lesssim r,$$

so $x \in B(a, r)$. Moreover, recall from our construction that all of these endpoints of intervals are actually in E , so the endpoint x in particular is an element of $B(a, r) \cap E$ distinct from a . Thus a is a limit point of E . (Note that technically, to show a is a limit point, we are supposed to show that any open set U containing a also contains a point of E distinct from a . We have only done this when $U = B(a, r)$. Can you explain why this is sufficient?)

- The interior of E is empty. To see this, suppose that $a \in E$ and consider an open ball $B(a, r)$ centered at a . As above, let n be big enough so that $3^{-n} \leq r$. Then, since E_n is the union of several disjoint closed intervals, let I_n be the interval containing a , and note that we must have $I_n \subseteq B(a, r)$. But then if we pick a point x in the middle third of I_n , then $x \notin E_{n+1}$, which means that $x \notin E$. But x is a point of $B(a, r)$, since all of I_n is contained in $B(a, r)$, so we see that the open ball $B(a, r)$ is not entirely contained in E . In other words, a cannot be an interior point of E .

You will have the opportunity to prove more properties of the Cantor set on problem set 3.

4 Sample Problems

Problem 1. Show that the Cantor set E is compact.

Hint. Do this by applying the Heine-Borel theorem. Boundedness should be easy: in fact, you should even be able to compute $\text{diam}(E)$ precisely. To see that E is closed, explain why each E_n is closed, and then explain why that implies that E is also closed.

Problem 2. Show that every point of the Cantor set E is a limit point of $[0, 1] \setminus E$.

Hint. This is basically the same statement as one of the properties we already proved about the Cantor set above. Which property? Why?