Compact Spaces

1 Open Covers

Let X be a metric space. An *open cover* for X is a set $\mathcal{U}$ of open subsets of X whose union is X.

Example 1.1. Consider $X := \mathbb{R}$ with the euclidean metric. Then the following are some examples of open covers of $\mathbb{R}$.

$$\begin{aligned}
&\mathcal{U}_1 := \{\mathbb{R}\}\\
&\mathcal{U}_2 := \{(-\infty, 1), (-1, \infty)\}\\
&\mathcal{U}_3 := \{B(n, 1) : n \in \mathbb{Z}\} = \{\dots, (-2, 0), (-1, 1), (0, 2), (1, 3), \dots\}\\
&\mathcal{U}_4 := \mathcal{U}_1 \cup \mathcal{U}_2 = \{\mathbb{R}, (-\infty, 1), (-1, \infty)\}
\end{aligned}$$

Given an open cover $\mathcal{U}$ of a metric space X, a *subcover* of $\mathcal{U}$ is a subset $\mathcal{U}' \subseteq \mathcal{U}$ whose union is still all of X.

Example 1.2. In example 1.1, $\mathcal{U}_1$ and $\mathcal{U}_2$ are both subcovers of $\mathcal{U}_4$. The only subcover of $\mathcal{U}_3$ is $\mathcal{U}_3$ itself, since if we remove $B(n, 1)$ from $\mathcal{U}_3$ for some integer n, then no other open subset inside $\mathcal{U}_3$ contains the integer n.

2 Compact Spaces

A metric space X is *compact* if every open cover has a finite subcover.

Remark 2.1. Like with connectedness, compactness is also a property of metric spaces. So, when someone says "let K be a compact subset of a metric space X," what is formally meant is that we take the metric on X, restrict it to a metric on K in order to regard K as a metric space in its own right, and then we require that this metric space K be compact.

Example 2.2. In example 1.2, we noted that $\mathcal{U}_3$ is an example of an open cover of $\mathbb{R}$ which has no nontrivial subcover. Since $\mathcal{U}_3$ is infinite, in particular we see that it has no finite subcover. Thus $\mathbb{R}$ is not compact.

Example 2.3. Consider $X := (0, 1)$ as a metric space by restricting the euclidean metric on $\mathbb{R}$. Then X is not compact either. For example, consider the open cover

$$\mathcal{U} := \{(1/n, 1) : n = 1, 2, \dots\}.$$

To see that this is an open cover of $(0,1)$, suppose we have any $r \in (0,1)$. Then by the archimedean property of $\mathbb{R}$, there exists some n such that $1/n \lneq r$, which means that $r \in (1/n, 1)$. Thus the union of all sets in $\mathfrak{U}$ is precisely $(0,1)$.

On the other hand, $\mathfrak{U}$ has no finite subcover. Indeed, suppose we have some finite subset $\mathfrak{U}' \subseteq \mathfrak{U}$, and let

$$n := \max\{j \in \mathbb{N} : (1/j, 1) \in \mathfrak{U}'\}.$$

Then $1/(n+1) \notin (1/j, 1)$ for all j such that $(1/j, 1) \in \mathfrak{U}'$, so $1/(n+1)$ is an element of $(0,1)$ which cannot be in the union over any finite subset of $\mathfrak{U}$. Thus $\mathfrak{U}$ has no finite subcover.

Example 2.4. Consider the set $X := \{0\} \cup \{1/n : n = 1, 2, \dots\}$ regarded as a metric space by restricting the euclidean metric on $\mathbb{R}$. This is compact. Indeed, suppose $\mathfrak{U}$ is an open cover. Then there exists some $U_0 \in \mathfrak{U}$ such that $0 \in U_0$. Since U_0 is open, there exists some positive real r such that $B(a, r) \subseteq U_0$. Furthermore, by the archimedean property of the reals, there exists some positive integer N such that $1/N \lneq r$. Now note that for any $n \geq N$, we have

$$1/n \leq 1/N \lneq r$$

which means that $1/n \in B(a, r) \subseteq U_0$. Now for each $i = 1, \dots, N-1$, there exists some open set $U_i \in \mathfrak{U}$ containing $1/i$, and then

$$\mathfrak{U}' := \{U_0, U_1, \dots, U_{N-1}\}$$

is a finite subcover of $\mathfrak{U}$.

Example 2.5. Consider $X := [0,1]$ as a metric space by restricting the euclidean metric on X. Then X is compact, but this is not so easy to prove. We will prove this a bit later after we have developed some more theory. For now, you should do problem 1.

Example 2.6. For some examples involving the discrete metric, see problem 2.

Lemma 2.7. *Let X be a metric space. If X is compact, then every infinite subset of X has a limit point in X.*

Proof. Let E be an infinite subset of X and suppose for a contradiction that E has no limit points in X. Fix a point $x \in X$. Then x is not a limit point of E, so there exists an open set U_x containing x such that U_x contains no points of E other than possibly x itself. In other words, if $x \in E$, then the only point of E contained in U_x is x itself, and if $x \notin E$, then U_x contains no points of E. Thus U_x contains at most one point of E.

Now consider

$$\mathfrak{U} := \{U_x : x \in X\}.$$

You should be able to see that $\mathfrak{U}$ is an open cover of X. But notice that it has no finite subcover. Indeed, if $\mathfrak{U}'$ is a finite subcover, then every $U \in \mathfrak{U}'$ contains at most one point of E, so the union over all sets in $\mathfrak{U}'$ can only contain finitely many points of E, but E is infinite so this union must miss some points of E. Thus we have contradicted compactness of X. ☐

Remark 2.8. The converse of lemma 2.6 is a problem for you in problem set 2. Also, you should do problem 3 below.

3 Bounded Spaces

Let X be a metric space. If X is nonempty, the *diameter* of X, denoted $\operatorname{diam}(X)$, is defined by

$$\operatorname{diam}(X) := \sup\{d(x,y) : x, y \in X\}.$$

We say that X is *bounded* if either X is empty or if $\operatorname{diam}(X)$ is finite. Equivalently, note that X is bounded if and only if there exists some real number R such that $d(x,y) \leq R$ for all $x, y \in X$. If X is not bounded, it is *unbounded*.

Example 3.1. In this example, regard all sets as metric spaces by restricting the euclidean metric on $\mathbb{R}$. Then $[0,1]$ and $(0,1)$ are both bounded of diameter 1, but $\mathbb{Z}$, $(0,\infty)$ and $\mathbb{R}$ are unbounded.

Lemma 3.2. *Let X be a compact metric space. Then X is bounded.*

The proof of lemma 3.2 will be a problem on problem set 3.

4 Compactness of Subsets

Even though compactness is a property of metric spaces, it is still useful to be able to identify when a subset E of a metric space X will be compact when E is regarded as a metric space in its own right by restricting the metric on X.

Lemma 4.1. *Let X be a metric space and let E be a subset of X. Suppose further that U is a subset of E. Then U is open as a subset of E if and only if there exists a subset $V \subseteq X$ such that V is open as a subset of X and $U = E \cap V$.*

Proof. Suppose there exists an open subset V of X such that $U = V \cap E$ and suppose $a \in U$. Since $a \in V$ and V is open in X, there exists some positive real r such that

$$\{x \in X : d(a,x) \lneq r\} \subseteq V.$$

This means that

$$\{x \in E : d(a,x) \lneq r\} = \{x \in X : d(a,x) \lneq r\} \cap E \subseteq V \cap E = U$$

so a is an interior point of U as a subset of E. Thus U is open inside E.

Conversely, suppose U is open as a subset of E. Then for every $a \in U$, there exists some positive integer r_a such that

$$\{x \in E : d(a,x) \lneq r_a\} \subseteq U. \tag{1}$$

Now consider

$$V := \bigcup_{a \in U} \{x \in X : d(a,x) \lneq r_a\}.$$

Then V is a union of open balls of X, so it is open since arbitrary unions of open sets are open. It is also clear that $U \subseteq V$, so we immediately have $U \subseteq V \cap E$. On the other hand, suppose

$x \in V \cap E$. Since $x \in V$, there exists some $a \in U$ such that $d(a, x) \lneq r_a$. But $x \in E$ also, so the fact that $d(a, x) \lneq r_a$ actually implies that $x \in U$ by (1) above. Thus $U = V \cap E$ as we wanted. □

Lemma 4.2. *Let X be a metric space and let E be a subset of X. Then E is a compact metric space if and only if, whenever we have a collection $\mathcal{U}$ of open subsets of X whose union contains E, there exists a finite subcollection $\mathcal{U}' \subseteq \mathcal{U}$ whose union still contains E.*

Proof. Suppose that E is a compact metric space and let $\mathcal{U}$ be a collection of open subsets of X whose union contains E. Then consider

$$\mathcal{V} := \{U \cap E : U \in \mathcal{U}\}.$$

By lemma 4.1, the elements of $\mathcal{V}$ are open subsets of E. Moreover, since E was contained in the union of all elements of $\mathcal{U}$, we can see that $\mathcal{V}$ is an open cover of E. But E is compact, so there exists a finite subcover $\mathcal{V}'$. In other words, we have

$$\mathcal{V}' = \{U_1 \cap E, \ldots, U_n \cap E\}$$

for some $U_1, \ldots, U_n \in \mathcal{U}$, and since the union of the elements of $\mathcal{V}'$ is E, we see that the union of the elements of

$$\mathcal{U}' := \{U_1, \ldots, U_n\}$$

contains E.

The proof of the converse is left as an exercise. Note that you'll need to use lemma 4.1 for the converse as well. If you get stuck, ask me about it in office hours! □

Lemma 4.3. *Let X be a compact metric space and F a closed subset of X. Then F is compact.*

Proof. We will use the criterion of lemma 4.2. Let $\mathcal{U}$ be a collection of open subsets of X whose union contains F. Then

$$\mathcal{V} := \mathcal{U} \cup \{X \setminus F\}$$

is an open cover of X, so, since X is compact, it has a finite subcover $\mathcal{V}'$. Let

$$\mathcal{U}' := \mathcal{V}' \setminus \{X \setminus F\}.$$

Then for any $x \in F$, we know that there exists some $V \in \mathcal{V}'$ such that $x \in V$, and $V \neq X \setminus F$ since $x \in F$, so $V \in \mathcal{U}'$ also. Thus F is contained in the union over all elements of $\mathcal{U}'$. Thus F is compact. □

Lemma 4.4. *Let X be a metric space and K a compact subset of X. Then K is closed inside X.*

Proof. Suppose $a \in X \setminus K$. For any point $x \in K$, let $r_x := d(a, x)$ and define $U_x := B(x, r_x/2)$ and $V_x := B(a, r_x/2)$, and notice that $U_x \cap V_x = \emptyset$. Indeed, if there existed some $y \in U_x \cap V_x$, then

$$d(a, x) = r_x = r_x/2 + r_x/2 \gneq d(a, y) + d(y, x)$$

which contradicts the triangle inequality. Now notice that

$$\mathcal{U} := \{U_x : x \in K\}$$

is a collection of open subsets of X whose union clearly contains K, so since K is compact, lemma 4.2 guarantees that there is a finite subcollection $\mathcal{U}' \subseteq \mathcal{U}$ whose union still contains K. In other words, there are finitely many points $x_1, \dots, x_n \in K$ such that

$$K \subseteq U_{x_1} \cup \cdots \cup U_{x_n}.$$

Notice that

$$X \smallsetminus K \supseteq X \smallsetminus (U_{x_1} \cup \cdots \cup U_{x_n}) = (X \smallsetminus U_{x_1}) \cap \cdots \cap (X \smallsetminus U_{x_n}) \supseteq V_{x_1} \cap \cdots \cap V_{x_n} =: V.$$

Since finite intersections of open sets are open, and V is open. Moreover V clearly contains a, so there is some open ball $B(a, r) \subseteq V \subseteq X \smallsetminus K$. Thus a is an interior point of $X \smallsetminus K$. Since a was arbitrary, this shows that $X \smallsetminus K$ is open.

Alternatively, to finish this proof a slightly different way, we can also observe that

$$V = B\left(a, \min\{r_{x_1}, \dots, r_{x_n}\}/2\right)$$

so V is itself an open ball around a contained entirely in $X \smallsetminus K$. □

5 Heine-Borel Theorem

If X is any metric space, and E is some compact subset, lemmas 4.4 and 3.2 together tell us that E must be bounded and closed inside X. In general, there can exist closed and bounded subsets of a metric space which are not compact: see problem 4. But this cannot happen in $\mathbb{R}^n$ with the euclidean metric.

Theorem 5.1 (Heine-Borel)**.** *Let X be a subset of $\mathbb{R}^n$ with the euclidean metric. Then X is compact if and only if X is bounded and closed inside $\mathbb{R}^n$.*

We will only prove this theorem when $n = 1$. The proof for general n is not much harder: it's just that the notation is harder to keep track of. Alternatively, there are also ways of deriving the statement for general n from the special case when $n = 1$. In any case, we need an intermediate result in order to prove this theorem.

Lemma 5.2 (Nested intervals theorem)**.** *If we have an interval $I_n := [a_n, b_n]$ inside $\mathbb{R}$ for each $n \in \mathbb{N}$ such that*

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots,$$

then $\bigcap_{n \in \mathbb{N}} I_n$ is nonempty.

Proof. Let $A := \{a_n : n \in \mathbb{N}\}$. Then b_n is an upper bound of A for all n, since

$$a_k \le a_{k+n} \le b_{k+n} \le b_n$$

for all k. Thus $x := \sup A \leq b_n$ for all n. Clearly we have $a_n \leq x$ for all n also. In other words, we have $x \in [a_n, b_n] = I_n$ for all n. □

Remark 5.3. Note that, if we somehow already knew that closed intervals were compact, then the nested intervals theorem 5.2 would be a consequence of the more general result of problem 5. But we're actually going to use the nested intervals theorem to prove that closed intervals are compact, so we do need to prove the above result separately in order to avoid circularity.

Proof of the Heine-Borel thoerem 5.1 when $n = 1$. Suppose X is bounded and closed inside $\mathbb{R}$. Since it is bounded, there exists some interval $[a, b]$ such that $X \subseteq [a, b]$. You should be able to verify for yourself that, since X is closed inside $\mathbb{R}$, it is also closed inside $[a, b]$. Hint: you might use lemma 4.1. Then, by lemma 4.3, it suffices to show that $[a, b]$ is compact.

To make notation a bit less confusing, let's go ahead and assume that $a = 0$ and $b = 1$. The general case is not any harder at all, or alternatively we will later learn how to derive the general case from this specific case. Suppose that $I_0 := [0, 1]$ is not compact. This means that there exists an open cover $\mathfrak{U}$ which contains no finite subcover. Now if $[0, 1/2]$ was contained in the union of finitely many elements of $\mathfrak{U}$, and $[1/2, 1]$ was also contained in the union of finitely many elements of $\mathfrak{U}$, then $[0, 1]$ would also be contained in the union of finitely many elements of $\mathfrak{U}$, which contradicts our assumption on $\mathfrak{U}$. This means that either $[0, 1/2]$ or $[1/2, 1]$ cannot be covered by finitely many elements of $\mathfrak{U}$. Let I_1 be one of the two of these smaller intervals which cannot be covered by finitely many elements of $\mathfrak{U}$.

We now repeat this process. We subdivide I_1 into two pieces, and one of the two pieces cannot be covered by finitely many elements of $\mathfrak{U}$, so we let I_2 be such a piece, and so on and so forth. The result is a sequence of intervals

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots,$$

each of which cannot be covered by finitely many elements of $\mathfrak{U}$, and such that $\operatorname{diam}(I_n) = 2^{-n}$ for all n. Then the nested intervals theorem 5.2 tells us that there exists some $\alpha \in \bigcap_{n \in \mathbb{N}} I_n$. We also know that $\alpha \in U$ for some $U \in \mathfrak{U}$. Since U is open, there exists some positive real number r such that $B(\alpha, r) \subseteq U$.

Now there exists some positive integer n that's big enough so that $2^{-n} \lneq r$. You should be able to derive this fact from the archimedean property of $\mathbb{R}$. This means that $I_n \subseteq B(\alpha, r)$, since for any $x \in I_n$, we have

$$d(\alpha, x) \leq \operatorname{diam}(I_n) = 2^{-n} \lneq r,$$

which means that $I_n \subseteq U$. This is a contradiction, since it should have been impossible to conver I_n with finitely many elements of $\mathfrak{U}$, but we have just covered I_n with one element of $\mathfrak{U}$. □

6 Sample Problems

Problem 1. Let $X := [0, 1]$ regarded as a metric space by restricting the euclidean metric on $\mathbb{R}$. For any ε such that $0 \lneq \varepsilon \leq 1$, the collection

$$\mathfrak{U} := \{(1/n, 1]\} \cup \{[0, \varepsilon)\}$$

is an open cover of X. Show directly that $\mathcal{U}$ has a finite subcover.

Hint. Note that the only element of $\mathcal{U}$ containing 0 is $[0, \varepsilon)$, so any subcover *must* contain $[0, \varepsilon)$. Then use the archimedean property of the reals to prove that we only need finitely many of the other elements of $\mathcal{U}$ in order to cover X.

Problem 2. Let X be a set with the discrete metric. Show that X is compact if and only if X is finite.

Problem 3. Give an example of a non-compact metric space X and an infinite subset E of X which has no limit point in X.

Problem 4. Give an example of a metric space X and a subset E which is bounded and closed inside X, but is not compact.

Hint. Try using the discrete metric.

Problem 5. Let X be a metric space and suppose K_n is a compact subset of X for all n such that

$$K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots .$$

Show that $K := \bigcap_{n \in \mathbb{N}} K_n$ is compact.

Reference. See Rudin, theorem 2.36 and its corollary for a proof.