# Connected Spaces

## 1 Connected Spaces

A metric space $X$ is *connected* if there exists no nonempty proper subset of $X$ which is both open and closed, and is *disconnected* otherwise.

**Remark 1.1.** This is a very technical remark, but I think it is useful to understand this right away and get your bearings straight because the same remark will apply when we discuss compactness. Notice that connectedness is a property of metric spaces, whereas openness and closedness are properties of subsets of metric spaces. It doesn't make sense to talk about metric spaces being open or closed: it only makes sense to talk about subsets of metric space being open or closed inside that metric space. It does make sense to talk about metric spaces being connected.

Now, even though connectedness is a property of metric spaces and not of subsets of a metric spaces, you will regularly hear people say things about "connected subsets of a metric space." Formally, this is a sleight of hand. When people say "let $E$ be a connected subset of a metric space $X$," what is actually meant is that we should take the metric on $X$, restrict it to a metric on $E$ so that $E$ itself becomes a metric space, and then that resulting metric space $E$ is required to be connected.

**Example 1.2.** Let $X$ be a set with more than two points and consider the discrete metric on $X$. Then $X$ is disconnected: indeed, if $x$ is some point in $X$, then $E := \{x\}$ is both open and closed, since all subsets of $X$ are open and closed, and $E$ is neither empty nor equal to the whole space.

**Example 1.3.** Consider $X := (-\infty, 0) \cup (0, \infty)$ with the metric obtained by restricting the euclidean metric on $\mathbb{R}$. This is disconnected as well. Consider the set $E := (0, \infty)$. This is clearly a nonempty and proper subset of $X$. In fact, it is also both open and closed inside $X$. To see that it is open inside $X$, notice that for any $a \in E$, we have

$$B(a, a) := \{x \in X : d(a, x) \leq a\} = (0, 2a) \subseteq E$$

so every $a \in E$ is an interior point of $E$. For basically the same reason, $(-\infty, 0)$ is also open inside $X$, but then since

$$X \smallsetminus E = (-\infty, E),$$

we conclude that $E$ is also closed inside $X$.

In fact, inside $\mathbb{R}$, connectedness means something very intuitive.

**Proposition 1.4.** *Let $X$ be a subset of $\mathbb{R}$. Then $X$ is a connected metric space if and only if we have $[x, y] \subseteq X$ whenever $x, y \in X$.*

*Proof.* Suppose that there exist $x, y \in X$ such that $[x, y] \not\subseteq X$. That means that there exists some $z \in [x, y]$ such that $z \notin X$. Now consider the set $U := [z, \infty) \cap X$. This is nonempty since $y \in U$ and proper since $x \notin U$. To see that $U$ is open inside $X$, notice that for any $a \in U$, if we set $r := d(a, z) = |a - z|$, then

$$B(a, r) := \{x \in X : d(a, x) \lneq r\} = (a - r, a + r) \cap X \subseteq U$$

since $x \in (a - r, a + r)$ means that $x \gneq a - r = z$. I will leave it for you to prove that the subset $(-\infty, z] \cap X$ is also open in $X$. The proof should be very similar to the proof that $U$ is open. But now notice that

$$X \setminus U = (-\infty, z] \cap X$$

so $U$ is also closed. Thus $X$ is disconnected.

Conversely, suppose that $X$ is disconnected. Let $U$ be a nonempty and proper subset of $X$ which is both open and closed. Since $U$ is nonempty, it contains some point $x$, and since it is proper, there is also some point $y$ in the complement $X \setminus U$. We can assume without loss of generality that $x \lneq y$, because if $x \gneq y$, then we simply replace $U$ with its complement $X \setminus U$, which will also be a nonempty proper open and closed subset of $X$. (If you aren't comfortable with the logic of the previous sentence, then you should just split up the proof into two cases, one where $x \lneq y$ and the other where $x \gneq y$. The proofs in both of these cases will look very similar.) Let us assume for a contradiction that $[x, y] \subseteq X$.

Let us define

$$a := \sup(U \cap [x, y]).$$

Since $a \in [x, y]$ (make sure you can explain why this is the case), we know that $a \in X$ since we have assumed that $[x, y] \subseteq X$. Moreover, it follows from the definition of $a$ as a supremum that $a$ is a limit point of $U$ in $X$ (make sure you can also explain why this is the case). Since $U$ is closed in $X$, we see that $a \in U$. Also, since $U$ is open, we know that $X \setminus U$ is closed, so by the same logic,

$$b := \inf((X \setminus U) \cap [a, y])$$

is actually an element of $X \setminus U$. We then have $a \lneq b$. Now pick any real number $z$ such that $a \lneq z \lneq b$. Then $z \in [x, y]$, but we claim that we cannot have $z \in X$, which gives us a contradiction. Indeed, if $z \in X$, then either $z \in U$ or $z \in X \setminus U$. If $z \in U$, then $z$ is an element of $U \cap [x, y]$ which is larger than $a$, contradicting the definition of $a$. Similarly, if $z \in X \setminus U$, then $z$ is an element of $(X \setminus U) \cap [a, y]$ which is smaller than $b$, which again contradicts the definition of $b$. $\square$

## 2 Sample Problems

**Problem 1.** Let $X$ be a metric space. Show that $X$ is disconnected if and only if there exist two disjoint nonempty open subsets $U$ and $V$ of $X$ such that $X = U \cup V$.

*Solution.* Suppose $X$ is disconnected. Then there is a nonempty proper subset $U$ of $X$ which is both open and closed. Let $V := X \setminus U$. Then $V$ is nonempty since $U \neq X$, and it is open since $U$ is closed, and clearly $U$ and $V$ are disjoint, and $X = U \cup V$ is clear too, so we are done.

Conversely if there exist such disjoint nonempty open subsets $U$ and $V$, then $U$ is also a proper subset since $V$ is nonempty, and $U$ is also closed since $X \setminus U = V$ is open, so $X$ is disconnected. $\square$

**Problem 2.** Let $X$ be a metric space. Show that $X$ is disconnected if and only if there exist two disjoint nonempty closed subsets $E$ and $F$ of $X$ such that $X = E \cup F$.

*Solution.* I will leave this as an exercise for you. The proof is almost identical to the proof in the solution of the previous problem. $\square$