Metric Spaces

1 Definition

A *metric* (sometimes also called a *distance function*) on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following axioms for all $x, y, z \in X$.

(M1) (Positivity) $d(x, y) \geq 0$, and $d(x, x) = 0$.

(M2) (Symmetry) $d(x, y) = d(y, x)$.

(M3) (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

(M4) If $d(x, y) = 0$, then $x = y$.

A *metric space* is a set X equipped with a metric d. (A function satisfying all of the axioms except (M4) is said to be a *pseudometric*, and a set together with a pseudometric is a *pseudometric space*, but we won't pursue this degree of generality any further.) See the accompanying PDF for many examples of metric spaces.

2 Open Subsets

Let X be a metric space. An *open ball* of some positive radius r about some point $a \in X$ is the set

$$B(a, r) := \{x \in X : d(a, x) \lneq r\}.$$

Let E be a subset of X. Then a point a is an *interior point* of E if there exists some positive real number r such that $B(a, r) \subseteq E$. The *interior* of E, denoted E°, is the set of interior points of E, and we say that E is an *open* subset of X if $E = E^\circ$. In other words, E is open if and only if every point of E is an interior point of E. If this terminology is to make any sense, open balls had better be open. Before proceeding any further, let us verify that this indeed the case.

Lemma 2.1. *Let X be a metric space. Then every open ball $B(a, r)$ is an open subset of X.*

Proof. Suppose $b \in B(a, r)$. Then $d(a, b) \lneq r$, so let $s := r - d(a, b)$ and consider $B(b, s)$. Then for any $x \in B(b, s)$, we have

$$d(a, x) \leq d(a, b) + d(b, x) = (r - s) + d(b, x) \lneq (r - s) + s = r$$

which means that $x \in B(a, r)$. In other words, $B(b, s) \subseteq B(a, r)$, so $B(a, r)$ is open. $\square$

Example 2.2. Inside $\mathbb{R}$ with the euclidean metric, the set $E_1 := [0, \infty)$ is not open since there is no open ball around 0 which is entirely contained in E_1. On the other hand, the set $E_2 := (0, \infty)$ is open. Indeed, for any $a \in E_2$, consider $B(a, a)$. Then for any $x \in B(a, a)$, we must have $x \gneq 0$, since if $x \leq 0$, then

$$|a - x| = a - x \geq a$$

which contradicts the choice of $x \in B(a, a)$. Thus $x \gneq 0$, so $x \in E$, which shows that $B(a, a) \subseteq E$. The same argument also shows that the interior E_1° of E_1 is equal to E_2.

Example 2.3. In any metric space X, the sets $\emptyset$ and X are both open. This is obvious.

Example 2.4. Openness depends on the ambient metric space. Let $X := [0, \infty)$. Then X is an open subset of itself, as we noted in the previous example, but it is not an open subset of $\mathbb{R}$, as we noted two examples prior. Stated differently, openness is a property of *subsets* of a metric space, rather than of metric spaces themselves.

Example 2.5. Consider $\mathbb{R}^2$ and the origin $\mathbf{0} \in \mathbb{R}^2$. Let us think about what happens to the open balls $B(\mathbf{0}, 1)$ as we change the metric we have in mind on $\mathbb{R}^2$.

- With the euclidean metric, we get an open circle shape.
- With the Manhattan metric, we get an open diamond shape.
- With the maximum metric, we get an open square shape.

Lemma 2.6. *Let X be a metric space.*

(a) Let I be an arbitrary set indexing a collection $(U_i)_{i \in I}$ of open subsets of X. Then $U := \bigcup_{i \in I} U_i$ is also an open subset of X.

(b) If U and V are open subsets of X, then $U \cap V$ is also an open subset of X.

Proof. For part (a), suppose $a \in U$. Then there exists some $i \in I$ such that $a \in U_i$. Since U_i is open, there exists some open ball $B(a, r)$ contained in U_i, which means that $B(a, r) \subseteq U$. Thus a is also an interior point of U, so U is open.

For part (b), suppose $a \in U \cap V$. Since U and V are both open, there exists some positive reals r and s such that $B(a, r) \subseteq U$ and $B(a, s) \subseteq V$. Let $t := \min\{r, s\}$. Then $B(a, t) \subseteq B(a, r) \subseteq U$ and $B(a, t) \subseteq B(a, s) \subseteq V$, so $B(a, t) \subseteq U \cap V$. Thus a is an interior point of $U \cap V$, so we conclude that $U \cap V$ is open. $\square$

Corollary 2.7. *If $U_1, \ldots, U_n$ are all open subsets of a metric space X, then $U_1 \cap \cdots \cap U_n$ is open.*

Example 2.8. Infinite intersections need not be open. For example, consider $X := \mathbb{R}$ with the euclidean metric and let $U_n := B(0, 1/n)$. Then U_n is open, but

$$\bigcap_{n \in \mathbb{N} \smallsetminus \{0\}} U_n = \{0\}$$

and this is not open. You should be able to prove both that $\bigcap U_n = \{0\}$ and that $\{0\}$ is not open.

Example 2.9. Let X be a set with the discrete metric. Then for any $a \in X$ we have

$$B(a, 1/2) = \{a\}$$

so the singleton set $\{a\}$ is open by lemma 2.1. But then, since arbitrary unions of open sets are open, we can immediately conclude that all subsets of X are open.

Lemma 2.10. *Let E be a subset of a metric space X. Then its interior E° is the largest subset of E which is open inside X.*

Proof. Let us first show that E° is open inside X. Suppose $a \in E^\circ$. Then there exists some open ball $B(a, r)$ entirely contained inside E. But then for every $x \in B(a, r)$, we know that x is an interior point of $B(a, r)$ by lemma 2.1, so there exists some ball $B(x, s) \subseteq B(a, r)$, which in turn implies that $B(x, s) \subseteq E$. This means that x is an interior point of E also, so $B(a, r) \subseteq E^\circ$. Thus every point of E° is an interior point of E°, so E° is open.

Next, suppose that U is some subset of E which is open inside X. Then for any $a \in U$, there exists some open ball $B(a, r) \subseteq U$, but $U \subseteq E$ so actually $B(a, r) \subseteq E$. This means that a is an interior point of E, so $a \in E^\circ$. Thus $U \subseteq E^\circ$. Thus E° is the largest subset of E which is open inside X. □

3 Sample Problems

Problem 1. Which of the following functions defines a metric on $\mathbb{R}$?

$$\begin{aligned}
d_1(x, y) &= (x - y)^2 \\
d_2(x, y) &= \sqrt{|x - y|} \\
d_3(x, y) &= \left|x^2 - y^2\right| \\
d_4(x, y) &= |x - 2y| \\
d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}
\end{aligned}$$

Solution. d_3 is not a metric because $d_3(1, -1) = 0$ even though $1 \neq -1$. d_4 is not a metric because $d_4(1, 0) = 1$ but $d_4(0, 1) = 2$. d_1 is not a metric because $d_1(2, 0) = 4$ is greater than $d_1(2, 1) + d_1(1, 0) = 1 + 1 = 2$, so the triangle inequality fails.

Both d_2 and d_5 are metrics. To see this, let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing and concave function such that $\varphi(0) = 0$, and consider the function $d_\varphi(x, y) = \varphi(|x - y|)$. For example, if we take $\varphi(t) = \sqrt{t}$ then $d_\varphi = d_2$, and if we take $\varphi(t) = t/(1 + t)$ then $d_\varphi = d_5$. We will prove that d_φ must be a metric. I will leave it to you to verify that (M1), (M2) and (M4) are satisfied. For the triangle inequality (M3), suppose we have $x, y, z \in \mathbb{R}$. Then, since φ is increasing,

$$d_\varphi(x, z) = \varphi(|x - z|) \leq \varphi(|x - y| + |y - z|).$$

Let ℓ be the function whose graph is the line going through the points $(0, 0)$ and $(|x - y| +$

$|y-z|, \varphi(|x-y|+|y-z|))$. Then

$$\varphi(|x-y|+|y-z|) = \ell(|x-y|+|y-z|) \leq \varphi(|x-y|+\varphi(|y-z|)$$

since the graph of the function φ must lie entirely above the secant line ℓ due to concavity of φ. This proves the triangle inequality for d_φ. $\square$

Problem 2. What is the interior of $\mathbb{Q}$ as a subset of $\mathbb{R}$ with the euclidean metric?

Solution. The interior is empty. Indeed, for any $a \in \mathbb{Q}$, suppose there existed some open ball $B(a,r) \subseteq \mathbb{Q}$. We showed last time that between the two real numbers a and $a+r$, there must exist some irrational number x. But clearly $x \in B(a,r)$, so this contradicts $B(a,r) \subseteq \mathbb{Q}$. Thus no point of $\mathbb{Q}$ is an interior point, so the interior is empty. $\square$