Real Numbers

1 Existence Theorem

Last time, we saw that $\mathbb{Q}$ is an ordered field, but it didn't have the supremum property. It turns out that there exists a larger ordered field which does have the supremum property, called the field of *real numbers*.

Theorem 1.1. *There exists an ordered field $\mathbb{R}$ with the supremum property. Moreover, it is unique up to unique isomorphism.*

It is also not important if you don't know precisely what "unique up to unique isomorphism" means: just know that, basically, there is only one ordered field which has the supremum property. Also, recall that any ordered field contains $\mathbb{Q}$. In particular, the field $\mathbb{R}$ contains $\mathbb{Q}$. Elements of $\mathbb{R} \smallsetminus \mathbb{Q}$ are said to be *irrational*.

We won't prove this theorem. There are many proofs. If you want to read a proof, you can find one in the appendix to chapter 1 in Rudin, or in chapter 1 section 6 of Ross. I find these constructions a bit messy: if we have time at the end of the semester, I may also sketch my preferred construction for you. In any case, we can derive many of the properties of $\mathbb{R}$ without knowing anything about the construction: we just need to use the fact that $\mathbb{R}$ is an ordered field with the supremum property.

2 Archimedean Property

Proposition 2.1. *If $x \in \mathbb{R}$, then there exists some natural number n such that $n \gneq x$.*

Proof. Suppose not. Then $n \leq x$ for all $n \in \mathbb{N}$, so x is an upper bound for $\mathbb{N}$. Thus $\mathbb{N}$ is a nonempty subset of $\mathbb{R}$ which is bounded above, so since $\mathbb{R}$ has the supremum property, $\mathbb{N}$ has a supremum. Let $\alpha := \sup \mathbb{N}$. Since $\mathbb{R}$ is an ordered field, we know that $1 \gneq 0$, so $\alpha - 1 \lneq \alpha$. Since α is the supremum of $\mathbb{N}$, $\alpha - 1$ cannot be an upper bound for $\mathbb{N}$, so there exists some $n \in \mathbb{N}$ such that $\alpha - 1 \lneq n$. But this implies that $\alpha \lneq n + 1 \in \mathbb{N}$, which is a contradiction. $\square$

Corollary 2.2. *If $x, y \in \mathbb{R}$ and $x \gneq 0$, then there exists some natural number n such that $nx \gneq y$.*

Proof. By the proposition, there exists some natural number n such that $n \gneq y/x$. Now just clear the denominator. $\square$

3 Density of the Rationals

We noted earlier that $\mathbb{R}$ contains $\mathbb{Q}$. In fact, $\mathbb{Q}$ is even dense in $\mathbb{R}$. What this means (at least for now) is the following.

Proposition 3.1. *If $x, y \in \mathbb{R}$ and $x \lneq y$, then there exists some $p \in \mathbb{Q}$ such that $x \lneq p \lneq y$.*

Proof. We want to show that there exists a rational number $p = m/n$, where $m, n \in \mathbb{Z}$ such that $x \lneq m/n \lneq y$. We can assume without loss of generality that n is positive. By clearing denominators, we see that it is equivalent to show that there exist integers m and $n \gneq 0$ such that

$$xn \lneq m \lneq yn.$$

Since $y \gneq x$, we know that $y - x \gneq 0$, so by corollary 2.2 there exists some natural number n such that

$$yn - xn = (y - x)n \gneq 1.$$

Now intuitively it is clear that since yn and xn are more than 1 apart from one another, there exists an integer m between the two of them, which is exactly what we need to prove. But we have to prove this formally. The idea is to pick m to be the smallest integer greater than xn, and then the fact that yn is more than 1 bigger than xn will automatically imply that m is less than yn.

By the archimedean property 2.1, there exists some integer $k \gneq \max\{|xn|, |yn|\}$. In other words,

$$-k \lneq xn \lneq yn \lneq k.$$

Consider the set $K := \{j \in \mathbb{Z} : xn \lneq j \leq k\}$. This set is finite since it can only contain integers between $-k$ and k, and it is nonempty since $k \in K$, so it has a minimum. Let $m := \min K$. We know that $xn \lneq m$ since $m \in K$. Also, by our choice of m, we know that $m - 1 \leq xn$, which means that

$$m \leq xn + 1 \lneq yn$$

where we have used the fact that $yn - xn \gneq 1$ for the last step. This shows that $xn \lneq m \lneq yn$, which is exactly what we wanted. $\square$

4 Existence of Roots

Proposition 4.1. *For every real $x \geq 0$ and every positive integer n, there exists a unique positive real number y such that $y^n = x$. This real number y, called the* nth root of x, *is denoted $\sqrt[n]{x}$ or $x^{1/n}$. When $n = 2$, we write $\sqrt{x}$ in place of $\sqrt[2]{x}$.*

Proof idea. Proving uniqueness is easy: if y_1 and y_2 are two distinct nth roots of x, then without loss of generality we have $y_1 \lneq y_2$, but this would imply that $y_1^n \lneq y_2^n$, even though both are supposed to be equal to x. Thus there can exist at most one nth root.

The idea of the proof of existence is easy: consider the set

$$E := \{t \in \mathbb{R} : t^n \leq x\}.$$

It isn't too difficult to show that E is nonempty and that it is bounded above, so by the supremum property it has a supremum. Let $y := \sup E$. We then "just" need to check that we actually do have $y^n = x$. The technical details involved in actually checking this are rather cumbersome, so we won't do it. If you would like to see details, check the proof of theorem 1.21 in Rudin. $\square$

5 Extended Reals

We define the *extended real number system* to be the set $\mathbb{R}$ together with two additional symbols ∞ and $-\infty$. We extend the order on $\mathbb{R}$ to this new set by defining

$$-\infty \leq x \leq \infty$$

for all $x \in \mathbb{R}$. Then ∞ is an upper bound for *every* nonempty subset of the extended reals, so every nonempty subset (not just the ones which are bounded above) has a supremum in the extended reals. More precisely, if E is a subset of $\mathbb{R}$ which is not bounded above, then $\sup E = \infty$ in the extended reals. Similar remarks apply for infimums.

6 Sample Problems

Problem 1. Show that $\sqrt{2+\sqrt{2}}$ is irrational.

Solution. Let $r = \sqrt{2+\sqrt{2}}$ and suppose that r is rational. Squaring both sides, we find that $r^2 = 2+\sqrt{2}$, which means that

$$\sqrt{2} = r^2 - 2.$$

Since r is rational, $r^2 - 2$ is also rational. But we have already proved that $\sqrt{2}$ is irrational, so this is a contradiction. $\square$

Problem 2. If r is a nonzero rational and x is irrational, show that $r + x$ and rx are irrational.

Solution. Let $r + x = s$ and $rx = t$. If s is rational, then $x = s - r$ would have to be rational as well since $\mathbb{Q}$ is a subfield of $\mathbb{R}$, which is a contradiction. Also, similarly, if t were rational, then $x = t/r$ would have to be rational as well, again since $\mathbb{Q}$ is a subfield of $\mathbb{R}$, and this is again a contradiction. $\square$

Problem 3. If $x, y \in \mathbb{R}$ and $x \lneq y$, show that there exists an irrational number z such that $x \lneq z \lneq y$.

Solution. Since $x \lneq y$, we also have $x + \sqrt{2} \lneq y + \sqrt{2}$. By density of the rationals, there exists some $r \in \mathbb{Q}$ such that

$$x + \sqrt{2} \lneq r \lneq y + \sqrt{2}.$$

Subtracting through by $\sqrt{2}$, we see that $x \lneq r - \sqrt{2} \lneq y$. But $-\sqrt{2}$ is irrational, so by problem 2, we see that $r - \sqrt{2}$ is irrational as well, and we are done. $\square$

Problem 4. Let $a \in \mathbb{R}$ and let $E := \{r \in \mathbb{Q} : r \leq a\}$. Show that $\sup E = a$.

Solution. Clearly a is an upper bound for E. To see that it is the supremum, suppose that b is some real number such that $b \lneq a$. By the density of the rationals, there exists some $r \in \mathbb{Q}$ such that $b \lneq r \lneq a$. In particular, $r \in E$ and r is larger than b, so b cannot be an upper bound for E. Thus $a = \sup E$. $\square$

Problem 5. Let E be a subset of $\mathbb{Q}$ which is bounded above and let α be the supremum of E when regarded as a subset of $\mathbb{R}$. Show that E has a supremum when regarded as a subset of $\mathbb{Q}$ if and only if $\alpha \in \mathbb{Q}$.

Solution. If $\alpha \in \mathbb{Q}$, then clearly E has a supremum when regarded as a subset of $\mathbb{Q}$ since α itself will be the supremum. Conversely, suppose E has a supremum $\beta \in \mathbb{Q}$ when regarded as a subset of $\mathbb{Q}$. Then β is an upper bound for E inside $\mathbb{R}$ as well, so we must have $\alpha \leq \beta$. If $\alpha \lneq \beta$, there exists some rational number r such that $\alpha \lneq r \lneq \beta$. But then $r \geq \alpha$, so r is also an upper bound for E, but r is also rational, so r is a smaller rational upper bound than β. This contradicts the choice of β. Thus $\alpha = \beta$, so $\alpha \in \mathbb{Q}$. $\square$

Problem 6. Show that $\mathbb{Q}$ does not have the supremum property.

Solution. Let $E := \{x \in \mathbb{Q} : x \leq \sqrt{2}\}$. By problem 4, we know that the supremum of E when regarded as a subset of $\mathbb{R}$ is $\sqrt{2}$. But we know that $\sqrt{2} \notin \mathbb{Q}$, so by problem 5, we conclude that E cannot have a supremum in $\mathbb{Q}$. $\square$