More on Integration

1 Fundamental Theorem of Calculus, Part I

Theorem 1.1. *If $f : [a, b] \to \mathbb{R}$ is integrable and $F : [a, b] \to \mathbb{R}$ is a continuous function such that $F'(x) = f(x)$ for all $x \in (a, b)$, then*

$$\int_a^b f = F(b) - F(a).$$

Proof. Fix $\varepsilon \gneq 0$. Then there exists a partition P such that

$$U(f, P) - L(f, P) \leq \varepsilon.$$

By applying the mean value theorem to the function F interval $[P_{k-1}, P_k]$, we find that there is some $x_k \in (P_{k-1}, P_k)$ such that

$$(P_k - P_{k-1})f(x_k) = F(P_k) - F(P_{k-1}).$$

Then

$$F(b) - F(a) = \sum_{k=1}^{n}(F(P_k) - F(P_{k-1})) = \sum_{k=1}^{n} f(x_k)(P_k - P_{k-1}).$$

This means that

$$L(f, P) \leq F(b) - F(a) \leq U(f, P).$$

But we also have

$$L(f, P) \leq \int_a^b f \leq U(f, P)$$

so combining these, we see that

$$\left| \int_a^b f - (F(b) - F(a)) \right| \leq \varepsilon.$$

Since ε is arbitrary, we are done. □

This theorem lets us calculate lots of integrals quickly, since we know how to calculate derivatives already.

Corollary 1.2 (Integration by parts)**.** *If u and v are continuous functions on $[a, b]$ which are* *differentiable on (a,b), and u' and v' are integrable on $[a,b]$, then*

$$\int_a^b uv' + \int_a^b u'v = u(b)v(b) - u(a)v(a).$$

Proof. Observe that $g = uv$ is integrable (we haven't proved this, but it's true: see theorem 6.13(a) in Rudin if you're worried about this). Then $g' = uv' + u'v$ by the product rule for derivatives, so

$$\int_a^b uv' + \int_a^b u'v = \int_a^b g' = g(b) - g(a) = u(b)u(a) - u(a)v(a). \qquad \square$$

2 Fundamental Theorem of Calculus, Part II

If $a \gneq b$, then we define

$$\int_a^b f := -\int_b^a f,$$

and we also define

$$\int_a^a f := 0.$$

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be integrable and define a function $F : [a,b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f.$$

Then F is continuous. Moreover, if f is continuous at $c \in (a,b)$, then F is differentiable at x_0 and $F'(c) = f(c)$.

Proof. Let $R := \|f\|_{\sup}$. Fix $\varepsilon \gneq 0$. Whenever $|x-y| \le \varepsilon/R$ with $x \le y$, observe that

$$|F(y) - F(x)| = \left|\int_x^y f\right| \le \int_x^y |f| \le \int_x^y R = R(y-x) \le \varepsilon.$$

This shows that F is continuous. (In fact, it even shows that it is uniformly continuous, but we knew that automatically anyway since F is a continuous function on a compact set).

Suppose f is continuous at $x_0 \in (a,b)$. Then for $x \ne c$, observe that

$$\frac{F(x) - F(c)}{x - c} = \frac{1}{x-c}\int_c^x f.$$

Observe also that

$$f(c) = \frac{1}{x-c}\int_c^x f(c),$$

so then, combining these,

$$\frac{F(x) - F(c)}{x - c} - f(c) = \frac{1}{x-c}\int_c^x (f - f(c)).$$

Fix $\varepsilon \gneq 0$. Since f is continuous at c, there exists $\delta \gneq 0$ such that $|x - c| \le \delta$ implies $|f(x) - f(c)| \le$

ε. Then

$$\left|\frac{F(x)-F(c)}{x-c}-f(c)\right| = \frac{1}{|x-c|}\left|\int_c^x (f-f(c))\right| \le \frac{1}{|x-c|}\left|\int_c^x |f-f(c)|\right| \le \varepsilon$$

which shows that

$$F'(c) = \lim_{x\to c}\frac{F(x)-F(c)}{x-c} = f(c). \qquad \square$$

3 Change of Variables

Theorem 3.1. *Let I and J be open intervals and suppose $u : J \to I$ is a continuously differentiable function. If $f : I \to \mathbb{R}$ is continuous, then*

$$\int_a^b (f\circ u)u' = \int_{u(a)}^{u(b)} f$$

for all $a, b \in J$.

Proof. Since f and u are both continuous, so is the composite $f \circ u$. Moreover u' is continuous also, so the product $(f\circ u)u'$ is also continuous, and therefore also integrable. Fix some $c \in I$ and define

$$F(x) = \int_c^x f.$$

Since f is continuous on I, we know that F is differentiable on I by theorem 2.1. Let $g = F \circ u$ and observe that

$$g'(x) = F'(u(x))u'(x) = f(u(x))u'(x)$$

for all $x \in J$ by the chain rule. In other words, $g' = (f\circ u)u'$. Thus, using theorem 1.1 for the second step,

$$\int_a^b (f\circ u)u' = \int_a^b g' = g(b)-g(a) = F(u(b)) - F(u(a)) = \int_c^{u(b)} f - \int_c^{u(a)} f = \int_{u(a)}^{u(b)} f. \qquad \square$$

4 Differentiation and Integration of Power Series

Lemma 4.1. *If $\sum a_n x^n$ has radius of convergence R, then both of the power series*

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \quad \textit{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

also have radius of convergence R.

Proof. First note that $\sum n a_n x^{n-1}$ has the same radius of convergence as $\sum n a_n x^n$, since both of these converge for the same values of x. Similarly $\sum a_n x^{n+1}/(n+1)$ and $\sum a_n x^n/(n+1)$ also have the same radius of convergence.

Now note that $R = 1/\alpha$ where $\alpha := \limsup |a_n|^{1/n}$. Then

$$\limsup_{n\to\infty} \sqrt[n]{n\,|a_n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \alpha$$

since $\lim n^{1/n} = 1$. Thus $\sum n a_n x^n$ also has radius of convergence $1/\alpha = R$. Similarly,

$$\limsup_{n\to\infty} \sqrt[n]{\frac{|a_n|}{n+1}} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \alpha$$

since $\lim(n+1)^{1/n} = 1$ also. Thus $\sum a_n x^n/(n+1)$ also has radius of convergence $1/\alpha = R$. □

For the remainder of this section, we suppose $\sum a_n x^n$ has radius of convergence $R \gneq 0$. We define the function $f : (-R, R) \to \mathbb{R}$ by $f(x) = \sum a_n x^n$. Also, we define the functions $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = a_n x^n$. Recall that f is continuous on $(-R, R)$, which in particular guarantees integrability, and also that the series of functions $\sum f_n$ converges uniformly to f on compact subsets of $(-R, R)$.

Lemma 4.2. *For all $x \in (-R, R)$,*

$$\int_0^x f = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

Proof. Let us assume that $x \geq 0$, and the case $x \leq 0$ is analogous. We know that the series of functions $\sum f_n$ converges uniformly on the compact set $[0, x]$. Thus

$$\int_0^x f = \lim_{n\to\infty} \int_0^x f_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_k \int_0^x t^k = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^{k+1}}{k+1} = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}.$$ □

Lemma 4.3. *The function f is differentiable on $(-R, R)$ and, for all $x \in (-R, R)$, we have*

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^n.$$

Proof. Consider the function $g : (-R, R) \to \mathbb{R}$ defined by $g(x) = \sum n a_n x^{n-1}$. Then, by lemma 4.2, we know that

$$\int_0^x g = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0$$

for all $x \in (-R, R)$. If we choose some $S \in (0, R)$, observe that for all $x \in [-S, S]$, we have

$$f(x) = \int_0^x g + a_0 = \int_{-S}^x g + \left(a_0 - \int_{-S}^0 g\right).$$

Since g is continuous on $[-S, S]$, theorem 2.1 guarantees that f is differentiable and that $f'(x) = g(x)$ for all $x \in (-S, S)$ (since the term in the parentheses above is just some constant). In other words, we have

$$f'(x) = \sum_{k=1}^{\infty} n a_n x^{n-1}$$

for all $x \in (-S, S)$. Since S is arbitrary, we have the same result for all $x \in (-R, R)$. □

5 Sample Problems

Problem 1. Let $f : [0, \infty) \to \mathbb{R}$ be a nonnegative and monotonically decreasing function. Then the series of real numbers

$$\sum_{n=0}^{\infty} f(n)$$

converges if and only if

$$\lim_{n\to\infty} \int_0^n f$$

converges. (This is called the "integral test," and it is sometimes useful for checking whether or not a series converges.)

Hint. For each $n \in \mathbb{N}$, consider the partition $P^{(n)} := \{0, 1, 2, \ldots, n\}$ of $[0, n]$.