Applications of Differentiation

1 L'Hôpital's Rule

Here's a theorem you probably remember from when you took calculus.

Theorem 1.1 (L'Hôpital's Rule)**.** *Let S be a nonempty connected open subset of $\mathbb{R}$ and suppose that $a = \sup S$ or $a = \inf S$. Furthermore, suppose that f and g are differentiable functions $S \to \mathbb{R}$, that g and g' are both nonzero on S, and that*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$$

for some extended real number L. If either

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \text{ or } \lim_{x \to a} |g(x)| = \infty,$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Proof. Let us go ahead and assume that $a = \inf S$, since the proof when $a = \sup S$ is analogous. We make the following two claims.

(A) If $-\infty \leq L \lneq \infty$ and $L \lneq L_1$, then there exists some $a_1 \in S$ such that

$$\frac{f(x)}{g(x)} \lneq L_1$$

for all $x \in (a, a_1)$.

(B) If $-\infty \lneq L \leq \infty$ and $L_2 \lneq L$, then there exists some $a_2 \in S$ such that

$$L_2 \lneq \frac{f(x)}{g(x)}$$

for all $x \in (a, a_2)$.

Given these claims, we can complete the proof as follows. If L is finite and $\varepsilon \gneq 0$, then we can apply claim (A) with $L_1 := L + \varepsilon$ and claim (B) with $L_2 := L - \varepsilon$ in order to find numbers $a_1, a_2 \in S$

satisfying the above properties. Then taking $a_0 := \min\{a_1, a_2\}$, we see that we have

$$\left|\frac{f(x)}{g(x)} - L\right| \lneq \varepsilon$$

for all $x \in (a, a_0)$. This says precisely that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

If $L = -\infty$, then claim (A) together with the fact that L_1 can be arbitrarily negative completes the proof, and if $L = \infty$, then claim (B) together with the fact that L_2 can be arbitrarily large completes the proof.

Thus it suffices to prove claims (A) and (B). Since the proof of (2) is very similar the proof of (A), we'll only prove (A). Suppose we have $L \lneq K \lneq L_1$. Then we know that there exists some $b \in S$ such that

$$\frac{f'(x)}{g'(x)} \lneq K$$

for all $x \in (a, b)$. Note that if x and y are distinct elements of (a, b) with $x \lneq y$, then the generalized mean value theorem (together with the fact that g is injective since g' never vanishes on S) shows that there is some $c \in (x, y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(c)}{g'(c)}$$

which means that

$$\frac{f(x) - f(y)}{g(x) - g(y)} \lneq K. \tag{1}$$

Now if we are in the situation that

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0,$$

then taking the limit as x tends towards a in equation (1) shows us that

$$\frac{f(y)}{g(y)} \leq K \lneq L_1$$

for all $y \in (a, b)$, so we can take $a_1 := b$ and we are done. Suppose on the other hand that

$$\lim_{x \to a} |g(x)| = \infty.$$

It is a fact that g' is either always positive or always negative on S. This follows from the "intermediate value theorem for derivatives" (see theorem 29.8 in Ross or theorem 5.12 in Rudin). We haven't proved this, and I don't really want to prove it just for this small step in the proof: notice, for example, that if we happen to know that g' is continuous, then this follows from the usual intermediate value theorem. Hopefully this omission is not too egregious.

Let us assume that g' is always negative on S, and the other case is analogous. Then the only way to have $\lim_{x\to a}|g(x)| = \infty$ is to have $\lim_{x\to a} g(x) = \infty$. Moreover, since g never vanishes on S, we know that we must actually have that g is positive on all of S. Since g' is negative, we know that g is decreasing on S, so $(g(x) - g(y))/g(x)$ is positive. Multiplying (1) by this quantity, we see that

$$\frac{f(x) - f(y)}{g(x)} \lneq K \cdot \frac{g(x) - g(y)}{g(x)}.$$

Rearranging, we find that

$$\frac{f(x)}{g(x)} \lneq K \cdot \frac{g(x) - g(y)}{g(x)} + \frac{f(y)}{g(x)} = K + \frac{f(y) - Kg(y)}{g(x)}.$$

Holding y fixed and letting x tend towards a, we see that the fraction on the right hand side tends towards 0. This means that there exists $b' \in S$ such that this fraction is less than $L_1 - K$ for all $x \in (a, b')$. Then if we let $a_1 := \min\{b, b'\}$, we see that for all $x \in (a, a_1)$, we have that

$$\frac{f(x)}{g(x)} \lneq L_1. \qquad \square$$

2 Taylor's Theorem

Let S be a connected open neighborhood of 0 and suppose a function $f : S \to \mathbb{R}$ and that f is differentiable at least $n - 1$ times on S for some positive integer n. We then define $R_n : S \to \mathbb{R}$ by

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k.$$

Note that if f is infinitely differentiable on S and $\lim R_n(x) = 0$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} k^k.$$

In other words, if f is infinitely differentiable and $\lim R_n(x) = 0$, then $f(x)$ is equal to its *Taylor series around 0 evaluated at x*. We want to understand more precisely when this happens.

Theorem 2.1 (Taylor)**.** *Let S be a connected open neighborhood of 0 and suppose $f : S \to \mathbb{R}$ is differentiable at least n times on S for some positive integer n. For every nonzero $x \in S$, there exists some a in between 0 and x such that*

$$R_n(x) = \frac{f^{(n)}(a)}{n!} x^n.$$

Proof. Fix a nonzero $x \in S$. Note that there exists a unique number M such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{M}{n!} x^n.$$

We are trying to show that $M = f^{(n)}(a)$ for some a between 0 and x. To prove this, consider the function $g : S \to \mathbb{R}$ given by

$$g(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k + \frac{M}{n!} t^n - f(t).$$

If you stare at the above formula for a minute, you'll see that for any nonnegative integer $k \leq n-1$, we have $g^{(k)}(0) = 0$. Moreover, we also have $g(x) = 0$ by our choice of M. Since $g(0) = g(x) = 0$, Rolle's theorem tells us that there exists some x_1 between 0 and x such that $g'(x_1) = 0$. Then, since $g'(0) = g'(x_1) = 0$, Rolle's theorem applied again tells us that there is an x_2 between 0 and x_1 such that $g''(x_2) = 0$. We keep going on this way, until we find an x_n such that $g^{(n)}(x_n) = 0$. But this means exactly that $M = f^{(n)}(x_n)$, so taking $a = x_n$ completes the proof. $\square$

Example 2.2. We have seen before that the alternating harmonic series $\sum (-1)^n/n$ converges, but we haven't yet calculated the value of its limit. Here is one way to do this, assuming familiar properties of the natural logarithm log. Define $f : (-1, \infty) \to \mathbb{R}$ by $f(x) = \log(1 + x)$. Then

$$\begin{aligned}
f'(x) &= (1+x)^{-1} \\
f''(x) &= -(1+x)^{-2} \\
f^{(3)} &= 2(1+x)^{-3} \\
&\vdots \\
f^{(n)} &= (-1)^{n+1}(n-1)!(1+x)^{-n}
\end{aligned}$$

This means that $f^{(n)} = (-1)^{n+1}(n-1)!$, so the Taylor series of f about 0 is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0 + \sum_{k=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!} x^n = \sum_{k=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

Notice that this is exactly the alternating harmonic series when $x = 1$. By Taylor's theorem, for each n, there exists some a_n between 0 and 1 such that

$$R_n(1) = \frac{f^{(n)}(a_n)}{n!} = \frac{(-1)^{n+1}}{(1+a_n)^n n},$$

which means that

$$|R_n(1)| = \frac{1}{(1+a_n)^n n} \leq \frac{1}{n}$$

for all n. This means that $\lim R_n(1) = 0$ by the squeeze theorem, so the alternating harmonic series (ie, the Taylor series for f evaluated at $x = 1$) converges to $f(1) = \log(2)$.

Example 2.3. One usually defines the exponential function using the power series $\sum x^n/n!$. I want to explain why one would think to write this formula down as an application of Taylor's theorem. Suppose I'm looking for a function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(x)$ for all $x \in \mathbb{R}$. Notice that if f is some such function, then $f + c$ will also be such a function for any constant c, so we're going to need to pin things down a bit more (as you might remember from solving differential equations in math 54 anyway). So, let's also say that we want this function f to satisfy $f(0) = 1$.

Notice that $f' = f$ means that $f'' = f' = f$, and so forth, so f must be infinitely differentiable. Moreover, since f is differentiable, it is also continuous in particular, so any compact interval $[-R, R]$, f must be bounded by some constant C. But then $f^{(n)} = f$ so actually we have

$$\|f^{(n)}\|_{\sup} = \|f\|_{\sup} \leq C$$

for all $n \in \mathbb{N}$, where the $\| - \|_{\sup}$ is taken on the interval $[-R, R]$. Thus problem 4 guarantees that $R_n(x) = 0$, which means that f is equal to its Taylor series on $[-R, R]$. But R is arbitrary, so f is actually equal to its Taylor series everywhere on $\mathbb{R}$. But we can compute the Taylor series, since we know that $f(0) = 1$, so $f'(0) = f(0) = 1$, so $f''(0) = f'(0) = 1$, and so forth, so

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

3 Sample Problems

Problem 1. Calculate

$$\lim_{x \to 0} \frac{1 - e^x}{xe^x + e^x - 1}.$$

Problem 2. Calculate

$$\lim_{x \to \infty} \frac{p(x)}{e^x}.$$

where p is a polynomial function.

Problem 3. Prove the following "baby version" of L'Hôpital's rule directly (without appealing to the general version). Let S be a neighborhood of a point $a \in \mathbb{R}$ and suppose f and g are functions $S \to \mathbb{R}$ such that f and g are differentiable at a, $f(a) = g(a) = 0$, and $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Hint. Observe that for all $x \in S \setminus \{a\}$, we have

$$\frac{f(x)}{g(x)} = \frac{f(x) - 0}{g(x) - 0} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}.$$

Problem 4. Let S be a connected open neighborhood of 0 and suppose $f : S \to \mathbb{R}$ is infinitely differentiable on S. If there exists a constant C such that $\|f^{(n)}\|_{\sup} \leq C$ for all $n \in \mathbb{N}$, show that

$$\lim_{n \to \infty} R_n(x) = 0.$$

Reference. Ross, corollary 31.4.

Problem 5. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x \gneq 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Show that f is infinitely differentiable, and that $f^{(n)}(0) = 0$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. *Remark.* Thus, the Taylor series of f around 0 is just constantly 0. Clearly f is not itself equal to 0, so this is an example of a function that is infinitely differentiable but is not equal to its Taylor series.

Reference. Ross, section 31, example 3.