Uniform Continuity

1 Uniform Continuity

Let $X$ and $Y$ be metric spaces and $f : X \to Y$ a continuous function. Then $f$ is uniformly continuous if for every $\varepsilon \geq 0$, there exists $\delta \geq 0$ such that $d_Y(f(x), f(x')) \leq \varepsilon$ whenever $d_X(x, x') \leq \delta$.

Remark 1.1. It might be helpful to observe the logical forms of the $\varepsilon$-$\delta$ characterization of continuity juxtaposed against the logical form of the definition of uniform continuity.

Continuity: $\forall \varepsilon \geq 0 \forall x \in X \exists \delta \geq 0 \forall x' \in X (x' \in B(x, \delta) \implies f(x') \in B(f(x), \varepsilon))$

Uniform continuity: $\forall \varepsilon \geq 0 \exists \delta \geq 0 \forall x \in X \forall x' \in X (x' \in B(x, \delta) \implies f(x') \in B(f(x), \varepsilon))$

The only difference is that the order of $\forall x \in X$ and $\exists \delta \geq 0$ are switched, but this matters. In the former, the $\delta$ depends on the point $x$. In the latter, the same $\delta$ works for all points $x$.

Example 1.2. Let $X := [1, \infty)$ and let $Y := \mathbb{R}$ and consider the function $f : X \to Y$ given by

$$f(x) = \frac{1}{x^2}.$$ 

This function is uniformly continuous. Before checking this, let us make a calculation.

$$f(x) - f(x') = \frac{x'^2 - x^2}{x^2 x'^2} = \frac{(x' - x)(x' + x)}{x^2 x'^2} = \left(\frac{x'}{x^2} + \frac{1}{x x'^2}\right) (x' - x).$$

Notice that $x, x' \in X$ means that $x, x' \geq 1$, so the parenthetical quantity is at most equal to 2. In other words,

$$|f(x) - f(x')| \leq 2 |x - x'|.$$ 

Now given any $\varepsilon \geq 0$, let $\delta := \varepsilon / 2$. Then for any $x, x' \in X$ such that $|x - x'| \leq \delta$, we see that

$$|f(x) - f(x')| \leq 2 |x - x'| \leq 2\delta = \varepsilon.$$ 

Lemma 1.3. Let $X$ and $Y$ be metric spaces and let $f : X \to Y$ be a uniformly continuous function. If $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $X$, then $(f(x_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in $Y$.

Proof. Fix $\varepsilon \geq 0$. Then by uniform continuity there exists $\delta \geq 0$ such that $d_Y(f(x), f(x')) \leq \varepsilon$ whenever $d_X(x, x') \leq \delta$. Since $(x_n)_{n \in \mathbb{N}}$ is Cauchy, there exists an $N$ such that for all $m, n \geq N$ we have $d_X(x_m, x_n) \leq \delta$. Then $d_Y(f(x_m), f(x_n)) \leq \varepsilon$ for all $m, n \geq N$, so $(f(x_n))_{n \in \mathbb{N}}$ is Cauchy. \qed
Example 1.4. Let \( X := (0, \infty) \) and \( Y := \mathbb{R} \) and consider the function \( f : E \to Y \) given by \( f(x) = 1/x \). Notice that the sequence \( (x_n)_{n \in \mathbb{N}} \) where \( x_n = 1/(n+1) \) is Cauchy in \( X \). But \( f(x_n) = n + 1 \) and clearly the sequence \( (n+1)_{n \in \mathbb{N}} \) is not Cauchy in \( Y \), so lemma 1.3 shows that \( f \) cannot be uniformly continuous.

Proposition 1.5. Let \( X \) be a metric space, \( E \) a dense subset, \( Y \) a complete metric space, and \( f : E \to Y \) a uniformly continuous function. Then there exists a unique continuous function \( \tilde{f} : X \to Y \) such that \( \tilde{f}(x) = f(x) \) for all \( x \in E \). Moreover, \( \tilde{f} \) is even uniformly continuous.

Proof. The uniqueness assertion will follow as a consequence of a problem that will be assigned on problem set 6. To show existence, we begin with some observations. First, suppose that \( (x_n) \) converges to \( a \), then \( (x_n) \) is a Cauchy sequence.

Next, suppose that \( (x_n) \) and \( (y_n) \) are both sequences in \( E \) converging to the same point \( a \). Then in fact we must have \( \lim f(x_n) = \lim f(y_n) \). To see this, consider the sequence \( (x_n') := (x_0, x_0', x_1, x_1', x_2, x_2', \ldots) \).

This is a sequence in \( E \) which clearly still converges to \( a \). Thus, by what we observed above, we know that \( \lim f(x_n') \) exists. Moreover \( (f(x_n))_{n \in \mathbb{N}} \) and \( (f(x_n'))_{n \in \mathbb{N}} \) are both subsequences of \( (f(x_n'))_{n \in \mathbb{N}} \), so they both converge to the same limit as well.

Now for any point \( a \in X \), we know that since \( E \) is dense in \( X \) there exists a sequence \( (x_n)_{n \in \mathbb{N}} \) in \( E \) such that \( \lim x_n = a \). We then define

\[
\tilde{f}(a) := \lim_{n \to \infty} f(x_n).
\]

This limit exists by our first observation, and it is independent of the choice of sequence \( (x_n)_{n \in \mathbb{N}} \) by our second observation. In particular, whenever \( a \in E \), we can simply take the constant sequence \( (x_n)_{n \in \mathbb{N}} \) in which \( x_n = a \) for all \( n \in \mathbb{N} \) and then we see clearly that \( \tilde{f}(a) = f(a) \).

We now need to show that \( \tilde{f} \) is uniformly continuous. (Technically, we first need to show that \( \tilde{f} \) is continuous, but uniform continuity implies continuity, so we don’t actually need to do this separately.) To see this, fix \( \varepsilon > 0 \). Since \( f \) is uniformly continuous, there exists \( \delta > 0 \) such that \( d_Y(f(x), f(x')) \leq \varepsilon/3 \) whenever \( x, x' \in E \) are points such that \( d_X(x, x') \leq \delta \).

Suppose \( a, a' \in X \) are points of \( X \) such that \( d_X(a, a') \leq \delta/3 \). Choose sequences \( (x_n)_{n \in \mathbb{N}} \) and \( (x_n')_{n \in \mathbb{N}} \) converging to \( a \) and \( a' \), respectively. Then there exists an \( N_0 \) such that \( x_n \in B_X(a, \delta/3) \) for all \( n \geq N \) and an \( N_0' \) such that \( x_n' \in B_X(a', \delta/3) \) for all \( n \geq N' \). Let \( M_0 = \max\{N_0, N_0'\} \). Then notice that for all \( n \geq M_0 \), we have

\[
d_X(x_n, x_n') \leq d_X(x_n, a) + d_X(a, a') + d_X(a', x_n') \leq \delta,
\]

which means that \( d_Y(f(x_n), f(x_n')) \leq \varepsilon/3 \).

Since \( \tilde{f}(a) = \lim f(x_n) \), there exists some \( N_1 \) such that \( d_Y(\tilde{f}(a), f(x_n)) \leq \varepsilon/3 \) for all \( n \geq N \). Similarly there exists \( N_1' \) such that \( d_Y(\tilde{f}(a'), f(x_n)) \leq \varepsilon/3 \) for all \( n \geq N_1' \). Then let \( M_1 = \max\{N_1, N_1', 2M_0\} \).
max\{M_0, N_1, N'_1\}. For \( n \geq M_1 \), observe that

\[
d_Y(\tilde{f}(a), \tilde{f}(a')) \leq d_Y(\tilde{f}(a), f(x_n)) + d_Y(f(x_n), f(x'_n)) + d_Y(f(x'_n), \tilde{f}(a')) \leq \varepsilon.
\]

In other words, we have just shown that whenever \( a \) and \( a' \) are elements of \( X \) whose distance is less than \( \delta/3 \), the distance between \( \tilde{f}(a) \) and \( \tilde{f}(a') \) is less than \( \varepsilon \). Thus \( \tilde{f} \) is uniformly continuous. \( \Box \)

**Example 1.6.** We can now prove that the function from example 1.4 is not uniformly continuous in a different way. Let \( X := [0, \infty) \), \( E := (0, \infty) \) and \( Y = \mathbb{R} \). Consider the function \( f : E \to Y \) given by \( f(x) = 1/x \). Then \( f \) cannot be uniformly continuous. Indeed, notice that \( E \) is dense in \( X \), so if \( f \) were uniformly continuous, there would exist a continuous function \( \tilde{f} : X \to \mathbb{R} \) such that \( \tilde{f}(x) = f(x) \) for all \( x \in E \). But clearly

\[
\lim_{x \to 0} \tilde{f}(x) = \lim_{x \to 0} f(x)
\]

does not exist.

## 2 Sample Problems

**Problem 1.** Let \( Y := \mathbb{R} \). For each of the following metric spaces \( X \) and functions \( f : X \to Y \), determine if \( f \) is uniformly continuous.

(a) \( X = (0, 3) \) and \( f(x) = 1/(x - 3) \).

(b) \( X = (3, \infty) \) and \( f(x) = 1/(x - 3) \).

(c) \( X = [4, \infty) \) and \( f(x) = 1/(x - 3) \).

*Hint for (c).* Observe that

\[
f(x) - f(x') = \frac{1}{x - 3} - \frac{1}{x' - 3} = \frac{(x - 3) - (x' - 3)}{(x - 3)(x' - 3)} = \frac{x - x'}{(x - 3)(x' - 3)}
\]

and that the smallest possible value of the denominator is 1.

**Problem 2.** Let \( X := \mathbb{Z} \) regarded as a metric space by restricting the euclidean metric on \( \mathbb{R} \). Show that any continuous function \( f : X \to Y \) into any metric space \( Y \) is uniformly continuous.