Uniform Continuity

1 Uniform Continuity

Let X and Y be metric spaces and $f : X \to Y$ a continuous function. Then f is *uniformly continuous* if for every $\varepsilon \gneq 0$, there exists $\delta \gneq 0$ such that $d_Y(f(x), f(x')) \lneq \varepsilon$ whenever $d_X(x, x') \lneq \delta$.

Remark 1.1. It might be helpful to observe the logical forms of the ε-δ characterization of continuity juxtaposed against the logical form of the definition of uniform continuity.

$$\begin{aligned} \text{Continuity} : \forall \varepsilon \gneq 0 \quad \forall x \in X \quad \exists \delta \gneq 0 \quad \forall x' \in X \quad (x' \in B(x,\delta) \implies f(x') \in B(f(x),\varepsilon)) \\ \text{Uniform continuity} : \forall \varepsilon \gneq 0 \quad \exists \delta \gneq 0 \quad \forall x \in X \quad \forall x' \in X \quad (x' \in B(x,\delta) \implies f(x') \in B(f(x),\varepsilon)) \end{aligned}$$

The only difference is that the order of $\forall x \in X$ and $\exists \delta \gneq 0$ are switched, but this matters. In the former, the δ depends on the point x. In the latter, the same δ works for all points x.

Example 1.2. Let $X := [1, \infty)$ and let $Y := \mathbb{R}$ and consider the function $f : X \to Y$ given by

$$f(x) = \frac{1}{x^2}.$$

This function is uniformly continuous. Before checking this, let us make a calculation.

$$f(x) - f(x') = \frac{x'^2 - x^2}{x^2 x'^2} = \frac{(x'-x)(x'+x)}{x^2 x'^2} = \left(\frac{x'+x}{x^2 x'^2}\right)(x'-x) = \left(\frac{1}{x^2 x'} + \frac{1}{x x'^2}\right)(x'-x).$$

Notice that $x, x' \in X$ means that $x, x' \geq 1$, so the parenthetical quantity is at most equal to 2. In other words,

$$|f(x) - f(x')| \leq 2\,|x - x'|\,.$$

Now given any $\varepsilon \gneq 0$, let $\delta := \varepsilon/2$. Then for any $x, x' \in X$ such that $|x - x'| \lneq \delta$, we see that

$$|f(x) - f(x')| \leq 2\,|x - x'| \lneq 2\delta = \varepsilon.$$

Lemma 1.3. *Let X and Y be metric spaces and let $f : X \to Y$ be a uniformly continuous function. If $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in X, then $(f(x_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in Y.*

Proof. Fix $\varepsilon \gneq 0$. Then by uniform continuity there exists $\delta \gneq 0$ such that $d_Y(f(x), f(x')) \lneq \varepsilon$ whenever $d_X(x, x') \lneq \delta$. Since $(x_n)_{n \in \mathbb{N}}$ is Cauchy, there exists an N such that for all $m, n \geq N$ we have $d_X(x_m, x_n) \lneq \delta$. Then $d_Y(f(x_m), f(x_n)) \lneq \varepsilon$ for all $m, n \geq N$, so $(f(x_n))_{n \in \mathbb{N}}$ is Cauchy. □

Example 1.4. Let $X := (0, \infty)$ and $Y := \mathbb{R}$ and consider the function $f : E \to Y$ given by $f(x) = 1/x$. Notice that the sequence $(x_n)_{n\in\mathbb{N}}$ where $x_n = 1/(n+1)$ is Cauchy in X. But $f(x_n) = n+1$ and clearly the sequence $(n+1)_{n\in\mathbb{N}}$ is not Cauchy in Y, so lemma 1.3 shows that f cannot be uniformly continuous.

Proposition 1.5. *Let X be a metric space, E a dense subset, Y a complete metric space, and $f : E \to Y$ a uniformly continuous function. Then there exists a unique continuous function $\widetilde{f} : X \to Y$ such that $\widetilde{f}(x) = f(x)$ for all $x \in E$. Moreover, $\widetilde{f}$ is even uniformly continuous.*

Proof. The uniqueness assertion will follow as a consequence of a problem that will be assigned on problem set 6. To show existence, we begin with some observations. First, suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence in E converging to some point $a \in X$. Then in particular $(x_n)_{n\in\mathbb{N}}$ is Cauchy, so lemma 1.3 guarantees that $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. But Y is complete, so $\lim f(x_n)$ exists.

Next, suppose that $(x_n)_{n\in\mathbb{N}}$ and $(x'_n)_{n\in\mathbb{N}}$ are both sequences in E converging to the same point $a \in X$. Then in fact we must have $\lim f(x_n) = \lim f(x'_n)$. To see this, consider the sequence

$$(x''_n)_{n\in\mathbb{N}} = (x_0, x'_0, x_1, x'_1, x_2, x'_2, \dots).$$

This is a sequence in E which clearly still converges to a. Thus, by what we observed above, we know that $\lim f(x''_n)$ exists. Moreover $(f(x_n))_{n\in\mathbb{N}}$ and $(f(x'_n))_{n\in\mathbb{N}}$ are both subsequences of $(f(x''_n))_{n\in\mathbb{N}}$, so they both converge to the same limit as well.

Now for any point $a \in X$, we know that since E is dense in X there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that $\lim x_n = a$. We then define

$$\widetilde{f}(a) := \lim_{n\to\infty} f(x_n).$$

This limit exists by our first observation, and it is independent of the choice of sequence $(x_n)_{n\in\mathbb{N}}$ by our second observation. In particular, whenever $a \in E$, we can simply take the constant sequence $(x_n)_{n\in\mathbb{N}}$ in which $x_n = a$ for all $n \in \mathbb{N}$ and then we see clearly that $\widetilde{f}(a) = f(a)$.

We now need to show that $\widetilde{f}$ is uniformly continuous. (Technically, we first need to show that $\widetilde{f}$ is continuous, but uniform continuity implies continuity, so we don't actually need to do this separately.) To see this, fix $\varepsilon \gneq 0$. Since f is uniformly continuous, there exists $\delta \gneq 0$ such that $d_Y(f(x), f(x')) \lneq \varepsilon/3$ whenever $x, x' \in E$ are points such that $d_X(x, x') \lneq \delta$.

Suppose $a, a' \in X$ are points of X such that $d_X(a, a') \lneq \delta/3$. Choose sequences $(x_n)_{n\in\mathbb{N}}$ and $(x'_n)_{n\in\mathbb{N}}$ converging to a and a', respectively. Then there exists an N_0 such that $x_n \in B_X(a, \delta/3)$ for all $n \geq N$ and an N'_0 such that $x'_n \in B_X(a', \delta/3)$ for all $n \geq N'$. Let $M_0 = \max\{N_0, N'_0\}$. Then notice that for all $n \geq M_0$, we have

$$d_X(x_n, x'_n) \leq d_X(x_n, a) + d_X(a, a') + d_X(a', x'_n) \lneq \delta$$

which means that $d_Y(f(x_n), f(x'_n)) \lneq \varepsilon/3$.

Since $\widetilde{f}(a) = \lim f(x_n)$, there exists some N_1 such that $d_Y(\widetilde{f}(a), f(x_n)) \lneq \varepsilon/3$ for all $n \geq N$. Similarly there exists N'_1 such that $d_Y(\widetilde{f}(a'), f(x_n) \lneq \varepsilon/3$ for all $n \geq N'_1$. Then let $M_1 =$

$\max\{M_0, N_1, N_1'\}$. For $n \geq M_1$, observe that

$$d_Y(\widetilde{f}(a), \widetilde{f}(a')) \leq d_Y(\widetilde{f}(a), f(x_n)) + d_Y(f(x_n), f(x_n')) + d_Y(f(x_n'), \widetilde{f}(a')) \lneq \varepsilon.$$

In other words, we have just shown that whenever a and a' are elements of X whose distance is less than $\delta/3$, the distance between $\widetilde{f}(a)$ and $\widetilde{f}(a')$ is less than ε. Thus $\widetilde{f}$ is uniformly continuous. $\square$

Example 1.6. We can now prove that the function from example 1.4 is not uniformly continuous a different way. Let $X := [0, \infty), E := (0, \infty)$ and $Y = \mathbb{R}$. Consider the function $f : E \to Y$ given by $f(x) = 1/x$. Then f cannot be uniformly continuous. Indeed, notice that E is dense in X, so if f were uniformly continuous, there would exist a continuous function $\widetilde{f} : X \to \mathbb{R}$ such that $\widetilde{f}(x) = f(x)$ for all $x \in E$. But clearly

$$\lim_{x \to 0} \widetilde{f}(x) = \lim_{x \to 0} f(x)$$

does not exist.

2 Sample Problems

Problem 1. Let $Y := \mathbb{R}$. For each of the following metric spaces X and functions $f : X \to Y$, determine if f is uniformly continuous.

(a) $X = (0, 3)$ and $f(x) = 1/(x - 3)$.

(b) $X = (3, \infty)$ and $f(x) = 1/(x - 3)$.

(c) $X = [4, \infty)$ and $f(x) = 1/(x - 3)$.

Hint for (c). Observe that

$$f(x) - f(x') = \frac{1}{x-3} - \frac{1}{x'-3} = \frac{(x-3)-(x'-3)}{(x-3)(x'-3)} = \frac{x - x'}{(x-3)(x'-3)}$$

and that the smallest possible value of the denominator is 1.

Problem 2. Let $X := \mathbb{Z}$ regarded as a metric space by restricting the euclidean metric on $\mathbb{R}$. Show that any continuous function $f : X \to Y$ into any metric space Y is uniformly continuous.