# Continuous Functions

## 1 Notation

There will now be many metric spaces running around. So, when there can be ambiguity, we will write $d_X$ for the metric on a metric space $X$ and $B_X(a, r)$ for the open ball in $X$ of radius $r$ centered at $a \in X$. Also, while it is not strictly necessary, I find it is useful to be familiar with the word "neighborhood" at this point.

Let $X$ be a metric space and $a \in X$ a point. A *neighborhood* of $a$ is a subset $U$ of $X$ in which $a$ is an interior point. Note that $U$ need not itself be open in $X$. If it does happen to be open in $X$, we say that $U$ is an *open neighborhood* of $a$.

**Example 1.1.** $(-1, 1), [-1, 1], (-0.5, 72]$ and $(-1, 1.2] \cup (72, 100]$ are all examples of neighborhoods of the point $a = 0$ in the metric space $X = \mathbb{R}$ with the euclidean metric. Only the first one of these is an open neighborhood.

**Remark 1.2.** Rudin uses the word "neighborhood" for the concept we have been calling "open ball." Rudin's usage is not standard: the definition of "neighborhood" given above is what most people mean when they say "neighborhood."

## 2 Limits of Functions

Let $X$ and $Y$ be metric spaces and, for some fixed point $a \in X$, let $f : X \setminus \{a\} \to Y$ be a function. We say that

$$\lim_{x \to a} f(x) = b$$

if, whenever $U$ is a neighborhood of $b$ in $Y$, then $f^{-1}(U) \cup \{a\}$ is also a neighborhood of $a$ in $X$.

**Example 2.1.** Let $X = Y = \mathbb{R}$ and $a = 0$ and consider the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by the formula

$$f(x) = x^2.$$

Let's show that

$$\lim_{x \to 0} f(x) = 0.$$

Let $U$ be a neighborhood of $b := 0$ in $Y$. In other words, $b$ is an interior point of $U$, so there exists an open ball $B_Y(b, \varepsilon) \subseteq U$. Now consider the open ball $V := B_X(a, \sqrt{\varepsilon})$. Then for any

$x \in V \cap E \setminus \{a\} = V \setminus \{0\}$, we have

$$d(b, f(x)) = |f(x)| = \left|x^2\right| \lneq \left(\sqrt{\varepsilon}\right)^2 = \varepsilon$$

which means that $f(x) \in B_Y(b, \varepsilon) \subseteq U$. In other words, the entire open ball $V$ is contained in $f^{-1}(U) \cup \{a\}$, so $a$ is an interior point of $f^{-1}(U) \cup \{a\}$. In other words, $f^{-1}(U) \cup \{a\}$ is a neighborhood of $a$.

**Example 2.2.** Let $X$ and $Y$ be metric spaces, $a \in X$ a point and $f : X \setminus \{a\} \to Y$ a function such that

$$\lim_{x \to a} f(x) = b.$$

By definition, we know that if $U$ is a neighborhood of $b$, then $f^{-1}(U) \cup \{a\}$ is a neighborhood of $a$. But, even if $U$ is an *open* neighborhood, it need not be that $f^{-1}(U) \cup \{a\}$ is an *open* neighborhood of $a$. To see this, let $X = Y = \mathbb{R}$ and $a = 0$ again, and consider the function $f : X \setminus \{a\} \to Y$ given by

$$f(x) = \begin{cases} 1/2 & \text{if } x \in [2, 3] \\ x^2 & \text{if } x \notin [2, 3]. \end{cases}$$

It's easy to see that this function still has

$$\lim_{x \to a} f(x) = 0$$

just like in the previous example. But, consider the open neighborhood $U := B_X(a, 1)$. Then

$$f^{-1}(U) \cup \{a\} = (-1, 1) \cup [2, 3]$$

which is clearly not open. This is the reason for introducing the terminology "neighborhood of $a$" instead of just sticking with "open set containing $a$."

If we have metric spaces $X$ and $Y$, a point $a \in X$ and a function $f : X \to Y$, we can "forget" about the value of $f$ at $a$ and restrict $f$ to a function $X \setminus \{a\} \to Y$. This means that it makes sense to write

$$\lim_{x \to a} f(x) = b$$

even if $f$ is defined at $a$. Of course, this also means that the fact that

$$\lim_{x \to a} f(x) = b$$

is totally independent of the value of $f$ at $a$.

**Example 2.3.** Keep the same notation as in example 2.1, but now suppose that $f : X \to Y$ is actually defined everywhere by the formula

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

The same proof as in example 2.1 shows that we still have

$$\lim_{x \to 0} f(x) = 0$$

even though $f(x) \neq 0$. In fact, the above would be true no matter what the value of $f(x)$ was. It could be 0, or 1, or 100...

**Lemma 2.4.** *Let $X$ and $Y$ be metric spaces, $a \in X$ a point and $f : X \smallsetminus \{a\} \to Y$ a function. Suppose further that $a$ is a limit point of $X \smallsetminus \{a\}$. If there exist points $b$ and $b'$ in $B$ such that*

$$\lim_{x \to a} f(x) = b \text{ and } \lim_{x \to a} f(x) = b'$$

*then $b = b'$.*

*Proof.* Suppose $b \neq b'$. Then there exist disjoint neighborhoods $U$ and $U'$ of $b$ and $b'$, respectively. But then $f^{-1}(U) \cup \{a\}$ and $f^{-1}(U') \cup \{a\}$ are both neighborhoods of $a$, which means their intersection is a neighborhood of $a$ also. But notice that

$$(f^{-1}(U) \cup \{a\}) \cap (f^{-1}(U') \cup \{a\}) = f^{-1}(U \cap U') \cup \{a\} = \{a\}$$

since $U \cap U' = \emptyset$. This means that $\{a\}$ is a neighborhood of $a$, which means that $a$ cannot be a limit point of $X \smallsetminus \{a\}$. $\square$

**Lemma 2.5.** *Let $X$ and $Y$ be metric spaces, $a \in X$ a point, and $f : X \smallsetminus \{a\} \to Y$ a function. Then*

$$\lim_{x \to a} f(x) = b$$

*for some $b \in B$ if and only if*

$$\lim_{n \to \infty} f(x_n) = b$$

*for every sequence $(x_n)_{n \in \mathbb{N}}$ in $X \smallsetminus \{a\}$ which converges to $a$.*

*Proof.* For the "only if" direction, let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $X \smallsetminus \{a\}$ converging to $a$ and let $U$ be a neighborhood of $b$. Then $f^{-1}(U) \cup \{a\}$ is a neighborhood of $a$, which means that there exists an open ball $V$ such that $f(V \smallsetminus \{a\}) \subseteq U$. Since $\lim x_n = a$, there exists an $N \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq N$. Then $x_n \in V \smallsetminus \{a\}$, so $f(x_n) \in U$ for all $n \in \mathbb{N}$. This shows that $\lim f(x_n) = b$.

For the "if" direction, suppose that

$$\lim_{x \to a} f(x) \neq b.$$

This means that there exists a neighborhood $U$ containing $b$ for which $f^{-1}(U) \cup \{a\}$ is *not* a neighborhood of $a$. In other words, there does not exist any open ball around $a$ which is mapped by $f$ entirely into $U$. Consider the open ball $V_n := B_X(a, 1/n)$. Then $f$ does not map $V_n \smallsetminus \{a\}$ into $U$, so there exists some $x_n \in V_n \smallsetminus \{a\}$ such that $f(x_n) \notin U$. Now notice that $\lim x_n = a$, since for any $\varepsilon \gneq 0$, there exists $N$ such that $1/N \lneq \varepsilon$, and then, since $x_n \in V_n := B_X(a, 1/n)$, we have that

$$d(a, x_n) \lneq 1/n \leq 1/N \lneq \varepsilon.$$

But we cannot have $\lim f(x_n) \neq b$. Indeed, since $U$ is a neighborhood of $b$, there exists some open ball $B_Y(b,r) \subseteq U$, and we know that $f(x_n) \notin U$ for all $n$, so $f(x_n) \notin B_Y(b,r)$ for all $n$. In other words, $B_Y(b,r)$ is an example of an open set containing $b$ which contains none of the points of the sequence $(f(x_n))_{n\in\mathbb{N}}$. □

**Example 2.6.** Let $X = Y = \mathbb{R}$, suppose $a \in X$ and consider theh function $f : X \to Y$ given by $f(x) = 2x^2 + 1$. Let $(x_n)_{n\in\mathbb{N}}$ be any sequence in $X \smallsetminus \{a\}$ converging to $a$. Then

$$\lim_{x\to a} f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 2x_n^2 + 1 = 2a^2 + 1 = f(a).$$

Functions like this, where the limit at every point is equal to the value of the function itself, are called "continuous."

# 3 Continuity

Let $X$ and $Y$ be metric spaces and let $f : X \to Y$ be a function. Then $f$ is *continuous at* $a \in X$ if

$$\lim_{x\to a} f(x) = f(a).$$

Also we say that $f$ is *continuous* if it is continuous at every $a \in X$.

**Lemma 3.1.** *Let $X$ and $Y$ be metric spaces. A function $f : X \to Y$ is continuous at a point $a \in X$ if and only if, for every neighborhood $U$ of $f(a)$ in $Y$, the preimage $f^{-1}(U)$ is a neighborhood of $a$ in $X$.*

*Proof.* By definition, we have that $f$ is continuous at $a$ if and only if, for every neighborhood $U$ of $f(a)$, the preimage $(f^{-1}(U) \smallsetminus \{a\}) \cup \{a\}$ is a neighborhood of $a$. But notice that

$$(f^{-1}(U) \smallsetminus \{a\}) \cup \{a\} = f^{-1}(U).$$ □

**Example 3.2.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(a,b) = a + b$. We give $\mathbb{R}$ and $\mathbb{R}^2$ both the euclidean metric. We want to show that $f$ is continuous. Notice that $f$ being continuous depends only on what subsets of $\mathbb{R}^2$ are open and not on the specific metric we put on $\mathbb{R}^2$, so actually it is more convenient to put the maximum metric on $\mathbb{R}^2$ instead of the euclidean metric.

Fix a point $(a,b) \in \mathbb{R}^2$ and let $U$ be a neighborhood of $f(a,b) = a + b$. Then there exists some $\varepsilon \gneq 0$ such that $B_{\mathbb{R}}(f(a,b),\varepsilon) \subseteq U$. Now notice that

$$B_{\mathbb{R}^2}((a,b),\varepsilon/2) = B_{\mathbb{R}}(a,\varepsilon/2) \times B_{\mathbb{R}}(b,\varepsilon/2)$$

so for any $(x,y)$ in this open ball, we have

$$d(f(a,b), f(x,y)) = |(a+b) - (x+y)| \leq |a-x| + |b-y| \lneq \varepsilon.$$

In other words, the open ball $B_{\mathbb{R}^2}((a,b),\varepsilon/2)$ is entirely contained in $f^{-1}(U)$, so $f^{-1}(U)$ is a neighborhood of $(a,b)$. This completes the proof of continuity.

Now if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences in $\mathbb{R}$ converging to $x$ and $y$, respectively, then $((a_n, b_n))_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^2$ converging to $(a, b)$, which means that

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} f(a_n, b_n) = \lim_{(x,y)\to(a,b)} f(x, y) = f(a, b) = a + b.$$

Here, we used lemma 2.5 for the second equality and the fact that $f$ is continuous for the third. This proves the proposition concerning sums of limits we had left unproved before. For the other parts of that proposition, see problem 3.

**Corollary 3.3.** *Let $X$, $Y$ and $Z$ be metric spaces and let $f : X \to Y$ and $g : Y \to Z$ be functions. Suppose $f$ is continuous at $a \in X$ and $g$ is continuous at $f(a)$. Then the composite $h := g \circ f$ is continuous at $a$.*

*Proof.* Let $U$ be a neighborhood of $h(a) = g(f(a))$. Since $g$ is continuous at $f(a)$, we know that $g^{-1}(U)$ is a neighborhood of $f(a)$, and then since $f$ is continuous at $a$, we also know that $f^{-1}(g^{-1}(U))$ is a neighborhood of $a$. But notice that

$$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) = h^{-1}(U). \qquad \square$$

**Corollary 3.4.** *Let $X$ and $Y$ be metric spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is open in $X$ for every open subset $U$ of $Y$.*

*Proof.* Suppose first that $f$ is continuous and let $U$ be an open subset of $Y$. We want to show that every point $a \in f^{-1}(U)$ is an interior point of $f^{-1}(U)$. But notice that $a \in f^{-1}(U)$ means that $f(a) \in U$, so $U$ is a neighborhood of $f(a)$, so $f^{-1}(U)$ is a neighborhood of $a$. In other words, $a$ is an interior point of $f^{-1}(U)$.

Conversely, suppose that $f^{-1}(U)$ is open for every open subset $U \subseteq X$. We want to show that $f$ is continuous at every point $a \in X$, so let $U$ be any neighborhood of $f(a)$. Then there exists an open set $U'$ containing $f(a)$ such that $U' \subseteq U$, and by assumption we know that $f^{-1}(U')$ is open. Then there exists an open ball $B_X(a, r) \subseteq f^{-1}(U') \subseteq f^{-1}(U)$. Thus $a$ is an interior point of $f^{-1}(U)$, so $f^{-1}(U)$ is a neighborhood of $a$ and we are done. $\square$

**Example 3.5.** Let $X = Y = \mathbb{R}$ with the euclidean metric and consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \text{ and} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then for any irrational number $a$, notice that $B_Y(0, 1/2)$ is a neighborhood of $f(a) = 0$, but its preimage is the set of all irrational numbers, which is not a neighborhood of $a$. Similarly, for every rational number $a$, notice that $B_Y(1, 1/2)$ is a neighborhood of $f(a) = 1$, but its preimage is the set of all rational numbers, which is again not a neighborhood of $a$. Thus, this function $f$ is discontinuous everywhere!

# 4 Continuity and Connectedness

**Lemma 4.1.** *Let $X$ and $Y$ be metric spaces and let $f : X \to Y$ be a continuous function. If $X$ is connected, then its image $f(X)$ is also connected.*

*Proof.* Suppose that $U$ is a nonempty proper open and closed subset of $f(X)$. Since $U$ is open in $f(X)$, there exists an open set $V$ in $Y$ such that $U = V \cap f(X)$. Since $f$ is continuous, we know that $f^{-1}(V)$ is open in $X$, but observe that $f^{-1}(V) = f^{-1}(U)$ since $U = V \cap f(X)$. Similarly, since $f(X) \smallsetminus U$ is open in $f(X)$, we conclude that

$$f^{-1}(f(X) \smallsetminus U) = X \smallsetminus f^{-1}(U)$$

is also open in $X$. Thus $f^{-1}(U)$ is an open and closed subset of $X$, so, since $X$ is connected, we have either $f^{-1}(U) = \emptyset$ or $f^{-1}(U) = X$. But notice that since $U$ is a subset of $f(X)$, having $f^{-1}(U) = \emptyset$ forces $U = \emptyset$, which is a contradiction. On the other hand, having $f^{-1}(U) = X$ forces $U = f(X)$, which again is a contradiction. $\square$

**Theorem 4.2** (Intermediate value theorem)**.** *Suppose $a \lneq b$ are real numbers and $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(a) \lneq f(b)$. If $f(a) \leq y \leq f(b)$, then there exists $c \in [a, b]$ such that $f(c) = y$.*

*Proof.* We know that $[a, b]$ is connected, so its continuous image $f([a, b])$ is also connected by lemma 4.1. Since $f(a)$ and $f(b)$ are elements of $f([a, b])$ and this is a connected subset of $\mathbb{R}$, we know that the entire interval $[f(a), f(b)]$ is contained in $f([a, b])$. In particular, since $y \in [f(a), f(b)]$, we have that $y \in f([a, b])$ also, so there exists $c \in [a, b]$ such that $f(c) = y$. $\square$

# 5 Sample Problems

**Problem 1.** Let $X$ and $Y$ be metric spaces and let $f : X \to Y$ be a function. Show that $f$ is continuous at $a \in X$ if and only if, for all $\varepsilon \gneq 0$, there exists $\delta \gneq 0$ such that $d(a, x) \lneq \delta$ implies $d(f(a), f(x)) \lneq \varepsilon$.

**Problem 2.** Mimic the proof of example 3.2 in order to prove that each of the following functions is continuous, where $\mathbb{R}$ and $\mathbb{R}^2$ have the euclidean metric.

(a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -x$.

(b) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = xy$.

(c) $f : \mathbb{R} \smallsetminus \{0\} \to \mathbb{R}$ given by $f(x) = 1/x$.

**Problem 3.** Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that $f$ is continuous at 0. *Remark.* We haven't formally defined sin in this class yet, but in any case, you can use the fact that $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$.

**Problem 4.** Let $X$ be a set regarded as a metric space with the discrete metric.

(a) Show that any function $f : X \to Y$ into a metric space $Y$ is continuous.

(b) Is it also true that any function $f : Y \to X$ from any metric space $Y$ is also continuous?

*Hint for (b).* Let $X = \mathbb{R}$ with the discrete metric and let $Y = \mathbb{R}$ with the euclidean metric, and consider the function $f : X \to Y$ given by $f(x) = x$. Is this continuous?

**Problem 5.** Determine all points $a \in \mathbb{R}$ where the function $f : \mathbb{R} \to \mathbb{R}$ defined by the following formula is continuous.

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \text{ and} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

**Problem 6.** Let $X$, $Y$ and $Z$ be metric spaces and let $f : X \to Y$ and $g : X \to Z$ be continuous functions. Let $Y \times Z$ be given the product metric defined on problem 10 in problem set 2. In other words,

$$d_{X\times Y}((y, z), (y', z')) = \max\{d_Y(y, y'), d_Z(z, z')\}.$$

Show that the function $g : X \to Y \times Z$ defined by $g(x) = (f(x), g(x))$ is continuous.

*Proof sketch.* Let

$$U := B_{Y\times Z}((y, z), r) = B_Y(y, r) \times B_Z(z, r).$$

be an open ball in $Y \times Z$. Then

$$\begin{aligned} g^{-1}(U) &= \{x \in X : (f(x), gs(x)) \in U\} \\ &= \{x \in X : f(x) \in B_Y(y, r) \text{ and } g(x) \in B_Z(z, r)\} \\ &= f^{-1}(B_Y(y, r)) \cap f^{-1}(B_Z(z, r)). \end{aligned}$$

Since $f$ and $g$ are both continuous, this is an intersection of open subsets of $X$, so is itself open. This shows that $g^{-1}(U)$ is open for every open ball $U$ in $X \times Y$. Why does this imply that $g$ is continuous? □

**Problem 7.** Let $I := [0, 1]$ and let $f : I \to I$ be a continuous function. Show that there exists some $a \in I$ such that $f(a) = a$.

*Hint.* Consider the function $g : I \to \mathbb{R}$ given by $g(x) - x$ and use the intermediate value theorem.