Limit Superior and Limit Inferior

1 Limit Superior and Limit Inferior

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let E_0 be its set of subsequential limits in $\mathbb{R}$. We proved abstractly that E_0 must be a closed subset of $\mathbb{R}$. Now let

$$E := \left\{a \in \mathbb{R} \cup \{\pm\infty\} : a = \lim_{k\to\infty} x_{n_k} \text{ for some subsequence } (x_{n_k})_{k\in\mathbb{N}}\right\}.$$

Notice in particular that $E \supseteq E_0$, but E may also contain ∞ or $-\infty$ or possibly both. Since $(x_n)_{n\in\mathbb{N}}$ has a monotonic subsequence, and the limit of a monotonic subsequence is always defined, we see that E must be nonempty. We define the *limit superior* of E to be $\sup E$, and we write

$$\limsup_{n\to\infty} x_n := \sup E.$$

Dually, we define the *limit inferior* of E to be $\inf E$, and we write

$$\liminf_{n\to\infty} x_n := \inf E.$$

We then automatically have $\liminf x_n \leq \limsup x_n$, but these values need not be equal.

Example 1.1. For our favorite sequence $(1, 1/2, 1/3, \dots)$, we have $E = E_0 = \{0\}$. Thus its limit superior and limit inferior are both 0.

Example 1.2. The sequence $(x_n)_{n\in\mathbb{N}} = (1, -1, 2, -2, 3, -3, \dots)$ has an empty set of subsequential limits. In other words, $E_0 = \emptyset$. But $(1, 2, 3, \dots)$ is a subsequence and $\lim(1, 2, 3, \dots) = \infty$, and similarly $(-1, -2, \dots)$ is a subsequence and $\lim(-1, -2, -3, \dots) = -\infty$, so $E = \{\infty, -\infty\}$. Thus

$$\liminf_{n\to\infty} x_n = -\infty \text{ and } \limsup_{n\to\infty} x_n = \infty.$$

Example 1.3. The sequence $(x_n)_{n\in\mathbb{N}} = (0, 1, 0, 2, 0, 3, 0, 4, \dots)$ has $E_0 = \{0\}$ and $E = \{0, \infty\}$. Thus

$$\liminf_{n\to\infty} x_n = 0 \text{ and } \limsup_{n\to\infty} x_n = \infty.$$

The first result we will prove is that $\limsup x_n$ is itself the limit of some subsequence, and similarly with $\liminf x_n$. We keep the notation E_0 and E as above.

Lemma 1.4. *If E_0 is not bounded above, then $\infty \in E$. Similarly, if E_0 is not bounded below, then $-\infty \in E$.*

Proof. Suppose E_0 is not bounded above. To show that $\infty \in E$, we must construct a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\lim x_{n_k} = \infty$. We do this inductively. Let $n_0 := 0$. Inductively, suppose we have picked $n_0 \lneq n_1 \lneq \cdots \lneq n_k$. Since E_0 is not bounded above, we know that there exists a subsequential limit $a \gneq \max\{k, x_{n_k}\}$. The set $U := (\max\{k+1, x_{n_k}\}, \infty)$ is an open subset of $\mathbb{R}$ containing a, so there exists some $n_{k+1} \gneq n_k$ such that $x_{n_{k+1}} \in U$. This gives us a subsequence $(x_{n_k})_{k\in\mathbb{N}}$, and we now claim that

$$\lim_{k\to\infty} x_{n_k} = \infty.$$

Indeed, notice first that this subsequence is clearly monotonically increasing. For any R, there exists some integer K such that $K \geq R$, and then for all $k \geq K$ we have

$$x_{n_k} \geq x_{n_K} \gneq K \geq R.$$

This shows that $\infty \in E$. The proof when E_0 is not bounded below is analogous. □

Corollary 1.5. *We always have* $\limsup x_n \in E$ *and* $\liminf x_n \in E$.

Proof. If $\limsup x_n := \sup E \in \mathbb{R}$, then clearly $\sup E = \sup E_0$, but E_0 is closed in $\mathbb{R}$, so

$$\sup E = \sup E_0 \in E_0 \subseteq E.$$

Suppose $\sup E = \infty$. If E_0 is bounded above, then the only way to have $\sup E = \infty$ is to have $\infty \in E$, so we are done. If E_0 is not bounded above, then by lemma 1.4 we have $\infty \in E$ again and we are done. Finally, suppose $\sup E = -\infty$. We already noted that E is nonempty, so the only way to have $\sup E = -\infty$ is to have $E = \{-\infty\}$, so again we are done. The proof for infimums is analogous. □

2 Sample Problems

Problem 1. For any $a \gneq \limsup x_n$, show that there exists some $N \in \mathbb{N}$ such that $x_n \lneq a$ for all $n \geq N$. (Dually, for any $a \lneq \liminf x_n$, there exists some $N \in \mathbb{N}$ such that $x_n \gneq a$ for all $n \geq N$.)

Proof. Suppose for a contradiction that there exists some $a \gneq \limsup x_n := \sup E$ such that $x_n \geq a$ for infinitely many n. Pick out the subsequence of all terms which are greater than or equal to a, and then that subsequence has a monotonic subsequence, which we'll call $(x_{n_k})_{k\in\mathbb{N}}$. In other words, $(x_{n_k})_{k\in\mathbb{N}}$ is a monotonic subsequence of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \geq a$ for all k. This means that

$$a \leq \inf\{x_{n_k} : k \in \mathbb{N}\} \leq \sup\{x_{n_k} : k \in \mathbb{N}\}.$$

Thus, it doesn't matter whether this sequence is monotonically increasing or monotonically decreasing, what have is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$b := \lim_{k\to\infty} x_{n_k} \geq a \gneq \sup E.$$

This is a contradiction, because $b \in E$ by definition of E. □

Problem 2. Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence of nonzero numbers which converges to a positive number x. Then for any sequence $(y_n)_{n\in\mathbb{N}}$, show that

$$\limsup_{n\to\infty} x_n y_n = x \limsup_{n\to\infty} y_n.$$

Proof. By corollary 1.5, there exists a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ such that $\lim y_{n_k} = \limsup y_n$. Then $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of a convergent subsequence, so it also converges to x and

$$\lim x_{n_k} y_{n_k} = x \lim y_{n_k} = x \limsup y_n.$$

Thus $x \limsup y_n$ is a subsequential limit of $(x_n y_n)_{n\in\mathbb{N}}$, so $\limsup x_n y_n \geq x \limsup y_n$. Note that, depending on whether $\lim y_{n_k}$ is infinite or not, we have used two different theorems for the second equality.

For the reverse inequality, use corollary 1.5 again to choose a subsequence $(x_{n_k} y_{n_k})_{k\in\mathbb{N}}$ such that $\limsup x_n y_n = \lim x_{n_k} y_{n_k}$. Then $(y_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(y_n)_{n\in\mathbb{N}}$ and

$$y_{n_k} = \frac{x_{n_k} y_{n_k}}{x_{n_k}}.$$

The numerator converges to $\limsup x_n y_n$ and the denominator to the nonzero value x, so $\lim y_{n_k} = (\limsup x_n y_n)/x$, which shows that $\limsup y_n \geq (\limsup x_n y_n)/x$. Now multiply through by x. $\square$