Sequences in $\mathbb{R}$

1 General Facts

Proposition 1.1. *Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$ converging to x and y, respectively.*

(a) $\lim(x_n + y_n) = x + y$.

(b) $\lim(x_n y_n) = xy$.

(c) $\lim(-x_n) = -x$.

(d) If $x \neq 0$ and $x_n \neq 0$ for all n, then $(x_n^{-1})_{n\in\mathbb{N}}$ converges to x^{-1}.

We will prove proposition 1.1 later, because I think it's less confusing to prove when we have a bit more theory. For now, we'll just use this result a lot.

Lemma 1.2 (Squeeze theorem)**.** *Suppose $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences of real numbers and $0 \leq x_n \leq y_n$ for all n. If $\lim y_n = 0$, then $\lim x_n = 0$ also.*

The proof of the squeeze theorem 1.2 will be a problem on problem set 4.

2 Examples

Example 2.1. For any $p \gneq 0$, we have

$$\lim_{n\to\infty} \frac{1}{n^p} = 0.$$

To see this, let U be an open set containing 0 and let ε be such that $B(0,\varepsilon) \subseteq U$. Then, using the archimedean property, there exists some N such that $1/N \lneq \varepsilon^{1/p}$. Then for all $n \geq N$, we see that $1/n \leq 1/N \lneq \varepsilon^{1/p}$, so $1/n^p \lneq \varepsilon$. In other words, $1/n^p \in B(0,\varepsilon) \subseteq U$ for all $n \geq N$.

Example 2.2. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ where

$$x_n = \frac{n^4 - 2n + 7}{3n^4 + 2n^2 + 2}.$$

Then we can write

$$x_n = \frac{1 - \frac{2}{n^3} + \frac{7}{n^4}}{3 + \frac{2}{n^2} + \frac{2}{n^4}}$$

for all $n \geq 1$. Using example 2.1, we see that $\lim(1/n^4) = \lim(1/n^3) = \lim(1/n^2) = 0$. Using proposition 1.1, we see that

$$\lim_{n\to\infty}\left(1 - \frac{2}{n^3} + \frac{7}{n^4}\right) = 1.$$

Similarly,

$$\lim_{n\to\infty}\left(3 + \frac{2}{n^2} + \frac{2}{n^4}\right) = 3.$$

Since this limit is nonzero, we can again use proposition 1.1 to conclude that $\lim x_n = 1/3$.

Example 2.3. For any a such that $|a| \lneq 1$, we have

$$\lim_{n\to\infty} a^n = 0.$$

To see this, notice that $|a|^{-1} \gneq 1$, so $b := |a|^{-1} - 1$ is positive and $|a| = 1/(1+b)$. Furthermore, using the binomial theorem, we see that

$$(1+b)^n = 1 + nb + \ldots + b^n \gneq nb$$

which means that $|a|^n \lneq 1/nb$. Now for any $\varepsilon \gneq 0$, use the archimedean property to find $N \in \mathbb{N}$ such that $1/N \lneq b\varepsilon$ and note that for $n \geq N$, we have

$$|a|^n \lneq \frac{1}{nb} \leq \frac{1}{Nb} \lneq \varepsilon$$

so $a^n \in B(0, \varepsilon)$.

Example 2.4. We have

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

To see this, let $x_n = n^{1/n} - 1$. Then proposition 1.1 implies that it is sufficient to show that $\lim x_n = 0$. Notice that $(1 + x_n)^n = n$, which means that, for $n \geq 2$, we have

$$n = (1 + x_n)^n \geq 1 + nx_n + \binom{n}{2}x_n^2 \cdots + x_n^n \gneq \binom{n}{2}x_n^2 = \frac{n(n-1)}{2}x_n^2.$$

Rearranging, we find that

$$x_n \lneq \sqrt{\frac{2}{n-1}}.$$

For any open ball $B(0, \varepsilon)$, the archimedean property guarantees that there exists some N such that $1/(N-1) \lneq \varepsilon^2/2$. Then for all $n \geq N$, we have

$$0 \leq x_n \lneq \sqrt{\frac{2}{n-1}} \leq \sqrt{\frac{2}{N-1}} \lneq \varepsilon$$

which shows that $x_n \in B(0, \varepsilon)$ and completes the proof.

Example 2.5. For any $a \gneq 0$, we have

$$\lim_{n\to\infty} \sqrt[n]{a} = 1.$$

To see this, let $x_n = a^{1/n} - 1$. Then proposition 1.1 implies that it is sufficient to show that $\lim x_n = 0$. Now note that if $a \geq 1$, then for all $n \geq a$ we have $1 \leq a^{1/n} \leq n^{1/n}$, which means that

$$0 \leq x_n \leq n^{1/n} - 1.$$

As we saw in example 2.4, we have $\lim(n^{1/n} - 1) = 0$. By the squeeze theorem 1.2, it follows that $\lim x_n = 0$. Now suppose that $0 \lneq a \lneq 1$. Then $1/a \gneq 1$, so as we just saw, we have $\lim(1/a)^{1/n} = 1$. We now apply proposition 1.1(d).

3 Infinite Limits

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We say that

$$\lim_{n\to\infty} x_n = \infty$$

if, for every $R \in \mathbb{R}$, there exists some N such that $x_n \geq R$ for all $n \geq N$. In informal words, we are requiring that, no matter how big someone insists the terms of our sequence be, eventually the entire sequence is bigger than that. Similarly, we say that $\lim x_n = -\infty$ if $\lim(-x_n) = \infty$.

Note that, even if either of these conditions is satisfied, $(x_n)_{n\in\mathbb{N}}$ is *not* a convergent sequence: it is a special kind of non-convergent sequence. There are non-convergent sequences which do not tend to $\pm\infty$. For example, $(1, -1, 1, -1, \dots)$ is a non-convergent sequence which is not tending towards $\pm\infty$. Since sequences $(x_n)_{n\in\mathbb{N}}$ for which $\lim x_n = \pm\infty$ are *not* convergent, results about convergent sequences (proposition 1.1, for example) do *not* apply to such sequences. Here is a list of some properties of sequences with infinite limits.

Proposition 3.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $\lim x_n = \infty$, and let $(y_n)_{n\in\mathbb{N}}$ be some sequence with $\lim y_n = y$, where y is some value in the extended reals.*

(a) If $y \neq -\infty$, then $\lim(x_n + y_n) = \infty$.

(b) If $y \gneq 0$, then $\lim(x_n y_n) = \infty$.

(c) $\lim(x_n^{-1}) = 0$.

The proof of proposition 3.1 is left as an exercise. If you get stuck, you can find a proof of (b) in Ross, theorem 9.9, and of (c) in Ross, theorem 9.10. Theorem 9.10 in Ross is a bit stronger and proves a kind of converse to proposition 3.1(c) as well.

4 Monotonic Sequences

Recall that all convergent sequences are bounded. The converse is not always true: for example, the sequence $(1, -1, 1, -1, \dots)$ is bounded but clearly not convergent. A special class of sequences where the converse is true is the class of monotonic sequences. A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is *monotonically increasing* if $x_0 \leq x_1 \leq x_2 \leq \cdots$, and is *monotonically decreasing* if $x_0 \geq x_1 \geq x_2 \geq \cdots$. It is *monotonic* if it is either monotonically increasing or monotonically decreasing.

Lemma 4.1. *If $(x_n)_{n\in\mathbb{N}}$ is a monotonically increasing sequence in $\mathbb{R}$, then*

$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

Similarly, if $(x_n)_{n\in\mathbb{N}}$ is a monotonically decreasing sequence in $\mathbb{R}$, then

$$\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}.$$

Proof. Suppose $(x_n)_{n\in\mathbb{N}}$ is monotonically increasing and let $a := \sup E$ where $E := \{x_n : n \in \mathbb{N}\}$. Consider first the case when E is bounded above, so that $a \in \mathbb{R}$. For any open ball $B(a, \varepsilon)$, note that the number $a - \varepsilon$ is not an upper bound for E since it is strictly less than the supremum a, so there exists some N such that $a - \varepsilon \lneq x_N \leq a$. But then for all $n \geq N$, monotonicity implies that

$$a - \varepsilon \lneq x_N \leq x_n \leq a$$

so $x_n \in B(a, \varepsilon)$. Thus $\lim x_n = a$.

Now consider the case when E is not bounded above, so that $a = \infty$. Pick any real number R. Since E is not bounded above, there exists some N such that $x_N \geq R$. Then monotonicity guarantees that $x_n \geq x_N \geq R$ for all $n \geq N$, which shows that $\lim x_n = \infty$. $\square$

Corollary 4.2. *A monotonic sequence in $\mathbb{R}$ is convergent if and only if it is bounded.*

5 Sample Problems

Problem 1. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$ and let $(y_n)_{n\in\mathbb{N}}$ be a sequence converging to 0. Show that $(x_n y_n)_{n\in\mathbb{N}}$ converges to 0.

Hint. Let $R := \sup\{|x_n| : n \in \mathbb{N}\} + 1$. Then, for any $\varepsilon \gneq 0$, there exists an $N \in \mathbb{N}$ such that $y_n \in B(0, \varepsilon/R)$ for all $n \geq N$...

Problem 2. Let $s_0 := 1$ and then define

$$s_{n+1} = \left(\frac{n+1}{n+2}\right) s_n^2$$

for all $n \in \mathbb{N}$. Show that $\lim s_n = 0$.

Proof. Note that $s_0 \geq 1$, so by induction it is clear that $s_n \geq 0$ for all n. Moreover, it is similarly clear by induction that $s_n \leq 1$ for all n. We claim that $s_{n+1} \lneq s_n$ for all n. For $n = 0$, we compute directly that

$$s_1 = \left(\frac{1}{2}\right) 1^2 = \frac{1}{2} \lneq 1 = s_0.$$

Inductively, suppose $s_{n+1} \lneq s_n$. Then

$$s_{n+2} = \left(\frac{n+2}{n+3}\right) s_{n+1}^2 \leq s_{n+1}$$

using the fact that $n+2 \lneq n+3$ and that $s_{n+1} \leq 1$ which means that $s_{n+1}^2 \leq s_{n+1}$. This completes the induction. We have thus proved that $(s_n)_{n \in \mathbb{N}}$ is a monotonically decreasing sequence and that $0 \leq s_n$ for all $n \in \mathbb{N}$, so

$$\lim_{n \to \infty} s_n = \inf\{s_n : n \in \mathbb{N}\} \geq 0.$$

In particular, $(s_n)_{n \in \mathbb{N}}$ is a convergent sequence. Let $s := \lim s_n$. Notice that

$$\lim_{n \to \infty} \frac{n+1}{n+2} = 1$$

which means that, by taking lim in the recurrence equation $s_{n+1} = ((n+1)/(n+2))s_n^2$ and applying proposition 1.1, we find that $s = s^2$, which means that

$$0 = s^2 - s = s(s-1),$$

so either $s = 0$ or $s = 1$. But notice that $s_0 := 1$ and $s_{n+1} \lneq s_n$ for all n, so we cannot have $s = 1$. Thus we conclude that $s = 0$. $\square$