Compactness and Sequences

1 Compactness and Sequences

We are going to prove a very useful theorem that tells us that compactness is equivalent to imposing some conditions on sequences in a space. But, before we state the equivalent conditions, it will be convenient to make some definitions.

- A metric space X is *sequentially compact* if every sequence in X has a convergent subsequence.
- A metric space X is *totally bounded* if X can be covered by finitely many open balls of radius ε for every $\varepsilon \gneq 0$. You might remember seeing this property before: on problem set 2, there was a problem asking you to prove that, if every infinite subset of a metric space has a limit point, then that metric space is totally bounded. Problems 4 and 5 give further properties of totally bounded metric spaces.

Theorem 1.1. *Let X be a metric space. Then the following are equivalent.*

(a) X is compact.

(b) X is sequentially compact.

(c) X is complete and totally bounded.

Proof. For (a) implies (b), suppose X is compact and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Let $E := \{x_n : n \in \mathbb{N}\}$. If E is a finite set, then one of the values of E must repeat infinitely often in the sequence, and the subsequence of just those values is a convergent subsequence. The harder case is when E is infinite. But then E is an infinite subset of a compact space, so it has a limit point a. We want to show that a is also a subsequential limit of a, so let U be an open set containing a and fix a position $N \in \mathbb{N}$. Then U contains infinitely many elements of E since a is a limit point of E. In particular, it must contain some x_n with $n \geq N$ (because $x_0, \ldots, x_{N-1}$ is only finitely many elements of E, so, since U contains infinitely many elements of E, it must also contain some element other than one of these). Thus we can conclude (using the lemma we proved last time about subsequential limits) that a is a subsequential limit of $(x_n)_{n\in\mathbb{N}}$.

For (b) implies (c), suppose that X is sequentially compact. To see that X is complete, let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then sequential compactness guarantees that this has a convergent subsequence. As we proved last time, this guarantees that $(x_n)_{n\in\mathbb{N}}$ is itself convergent. Thus X is complete.

To see that X is totally bounded, suppose for a contradiction that X is not totally bounded, which means that there exists an ε such that finitely many open balls of radius ε cannot cover X. Pick some $x_0 \in X$. Then $B(x_0, \varepsilon)$ does not cover X, so there exists some $x_1 \in X \smallsetminus B(x_0, \varepsilon)$. Then the finite set of open balls $\{B(x_0, \varepsilon), B(x_1, \varepsilon)\}$ also does not cover X, so there exists some $x_2 \in X \smallsetminus (B(x_0, \varepsilon) \cup B(x_1, \varepsilon))$. You should attempt for yourself to formalize this inductive process. The end result is a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in X \smallsetminus (B(x_0, \varepsilon) \cup \cdots B(x_{n-1}, \varepsilon))$ for all n. Since X is sequentially compact, this sequence has a convergent subsequence, so let a be a subsequential limit. Consider the open ball $B(a, \varepsilon/2)$. Then there exists some m such that $x_m \in B(a, \varepsilon/2)$, and there exists some $n \geq m+1$ such that $x_n \in B(a, \varepsilon/2)$. We have used the lemma we proved last time about subsequential limits twice here. Then

$$d(x_m, x_n) = d(x_m, a) + d(a, x_n) \lneq \varepsilon,$$

which means that $x_n \in B(x_m, \varepsilon)$, even though we specifically chose

$$x_n \in X \smallsetminus (B(x_0, \varepsilon) \cup \cdots \cup B(x_{n-1}, \varepsilon)) \subseteq X \smallsetminus B(x_m, \varepsilon).$$

This gives us the contradiction we were seeking, so we can conclude that X is totally bounded.

Finally, we have (c) implies (a), which is the hardest part of the proof of this theorem. Suppose that X is complete and totally bounded and suppose for a contradiction that $\mathcal{U}$ is an open cover of X which does not have a finite subcover. Since X is totally bounded, by problem 5 we can cover X with finitely many subsets of diameter at most 1. Since $\mathcal{U}$ has no finite subcover, it must be that one of these subsets cannot be covered by finitely many elements of $\mathcal{U}$. Let A_0 be such a set. Now since X is totally bounded, we can apply problem 5 again to the set A_0 to cover it with finitely many subsets of diameter at most $1/2$. Proceeding in this way, we obtain a nested sequence of subsets

$$A_0 \supseteq A_1 \supseteq \cdots$$

such that $\operatorname{diam}(A_n) \leq 1/(n+1)$ and A_n cannot be covered by finitely many elements of $\mathcal{U}$.

The fact that A_n cannot be covered by finitely many elements of $\mathcal{U}$ implies in particular that A_n is nonempty, so pick some point $x_n \in A_n$ and consider the sequence $(x_n)_{n \in \mathbb{N}}$. This sequence is Cauchy. Indeed, for any $\varepsilon \gneq 0$, there exists some n such that $1/(N+1) \lneq \varepsilon$, and then for any $m, n \geq N$, we see that $x_m \in A_m \subseteq A_N$ and $x_n \in A_n \subseteq A_N$, so

$$d(x_m, x_n) \leq \operatorname{diam}(A_N) \leq 1/(N+1) \lneq \varepsilon.$$

Since X is complete, this sequence must be convergent. Let $a = \lim x_n$.

Since $\mathcal{U}$ is an open cover, there exists some $U \in \mathcal{U}$ such that $a \in U$, and some open ball $B(a, r) \subseteq U$. Then there exists some N such that $x_n \in B(a, r/2)$ for all $n \geq N$. Furthermore, there exists some N' such that $1/(N'+1) \lneq r/2$. Then for any $n \geq \max\{N, N'\}$, we have $A_n \subseteq B(a, r)$. Indeed, for any $y \in A_n$, we have

$$d(a, y) \leq d(a, x_n) + d(x_n, y) \lneq r/2 + \operatorname{diam}(A_n) \leq r/2 + 1/(n+1) \leq r/2 + 1/(N+1) \leq r.$$

Thus we have $A_n \subseteq U$ as well, which is a contradiction since we specifically chose A_n to not be coverable by finitely many elements of $\mathcal{U}$. $\square$

2 Sequences in $\mathbb{R}^n$

A sequence $(x_n)_{n\in\mathbb{N}}$ is *bounded* if the set $E := \{x_n : n \in \mathbb{N}\}$ is bounded. We proved last time that every Cauchy sequence is bounded. Moreover, it is also clear that any subsequence of a bounded sequence is again bounded. Of course, in general, bounded sequences need not be convergent. In general it need not even be that bounded sequences have convergent subsequences: see problem 6. But this latter phenomnenon cannot happen in $\mathbb{R}^n$.

Theorem 2.1 (Bolzano-Weierstrass)**.** *If a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^n$ is bounded, it has a convergent subsequence.*

Proof. Since the set $E := \{x_n : n \in \mathbb{N}\}$ is bounded, the closure $\bar{E}$ is as well, as we proved last time, which implies that it is compact by the Heine-Borel theorem. Then $(x_n)_{n\in\mathbb{N}}$ is a sequence in the compact set $\bar{E}$. Then $\bar{E}$ is sequentially compact by theorem 1.1, so it has a convergent subsequence. $\square$

Corollary 2.2. *$\mathbb{R}^n$ is complete.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}^n$. Then it is bounded, so the Bolzano-Weierstrass theorem 2.1 guarantees that it has a convergent subsequence. But, as we know, Cauchy sequences with convergent subsequences must themselves be convergent. $\square$

3 Sample Problems

Problem 1. Give an example of a sequence in $\mathbb{R}$ with no convergent subsequence.

Problem 2. Give an example of a sequence in the open interval $(0, 1)$ with no convergent subsequence.

Problem 3. We know that the open interval $(0, 1)$ is not compact. Thus, by the theorem, it must either not be complete or not be totally bounded, or both. Is it complete? Is it totally bounded?

Problem 4. Let X be a metric space.

(a) Show that, if X is totally bounded, then it is bounded.

(b) Give an example to show that X can be bounded without being totally bounded.

Hint. For (b), consider an infinte set with the discrete metric.

Problem 5. Let X be a totally bounded metric space and let E be a subset of X. Then, for any $\varepsilon \gneq 0$, there exist finitely many sets of diameter at most ε whose union equals E.

Proof. Since X is totally bounded, there exist finitely many open balls $B(x_1, \varepsilon/2), \ldots, B(x_n, \varepsilon/2)$ whose union equals X. Moreover, we know that $\operatorname{diam} B(x_i, \varepsilon/2) \leq \varepsilon$. Then $E_i := E \cap B(x_i, \varepsilon/2)$ are subsets of X of diameter at most ε, and clearly $E = E_1 \cup \cdots \cup E_n$. $\square$

Problem 6. Show that it is possible to have a bounded sequence in a metric space which has no convergent subsequences.

Hint. Let X be an infinite set with a discrete metric. Explain why the only convergent sequences in X are the ones which are eventually constant. Then choose some sequence in X all of whose points are distinct. Explain why this sequence must be bounded, and then explain why it cannot have any convergent subsequences.