1 Compactness and Sequences

We are going to prove a very useful theorem that tells us that compactness is equivalent to imposing some conditions on sequences in a space. But, before we state the equivalent conditions, it will be convenient to make some definitions.

- A metric space $X$ is **sequentially compact** if every sequence in $X$ has a convergent subsequence.

- A metric space $X$ is **totally bounded** if $X$ can be covered by finitely many open balls of radius $\varepsilon$ for every $\varepsilon \geq 0$. You might remember seeing this property before: on problem set 2, there was a problem asking you to prove that, if every infinite subset of a metric space has a limit point, then that metric space is totally bounded. Problems 4 and 5 give further properties of totally bounded metric spaces.

**Theorem 1.1.** Let $X$ be a metric space. Then the following are equivalent.

(a) $X$ is compact.

(b) $X$ is sequentially compact.

(c) $X$ is complete and totally bounded.

**Proof.** For (a) implies (b), suppose $X$ is compact and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $X$. Let $E := \{x_n : n \in \mathbb{N}\}$. If $E$ is a finite set, then one of the values of $E$ must repeat infinitely often in the sequence, and the subsequence of just those values is a convergent subsequence. The harder case is when $E$ is infinite. But then $E$ is an infinite subset of a compact space, so it has a limit point $a$. We want to show that $a$ is also a subsequential limit of $a$, so let $U$ be an open set containing $a$ and fix a position $N \in \mathbb{N}$. Then $U$ contains infinitely many elements of $E$ since $a$ is a limit point of $E$. In particular, it must contain some $x_n$ with $n \geq N$ (because $x_0, \ldots, x_{N-1}$ is only finitely many elements of $E$, so, since $U$ contains infinitely many elements of $E$, it must also contain some element other than one of these). Thus we can conclude (using the lemma we proved last time about subsequential limits) that $a$ is a subsequential limit of $(x_n)_{n \in \mathbb{N}}$.

For (b) implies (c), suppose that $X$ is sequentially compact. To see that $X$ is complete, let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence. Then sequential compactness guarantees that this has a convergent subsequence. As we proved last time, this guarantees that $(x_n)_{n \in \mathbb{N}}$ is itself convergent. Thus $X$ is complete.
To see that $X$ is totally bounded, suppose for a contradiction that $X$ is not totally bounded, which means that there exists an $\varepsilon$ such that finitely many open balls of radius $\varepsilon$ cannot cover $X$. Pick some $x_0 \in X$. Then $B(x_0, \varepsilon)$ does not cover $X$, so there exists some $x_1 \in X \setminus B(x_0, \varepsilon)$. Then the finite set of open balls $\{B(x_0, \varepsilon), B(x_1, \varepsilon)\}$ also does not cover $X$, so there exists some $x_2 \in X \setminus (B(x_0, \varepsilon) \cup B(x_1, \varepsilon))$. You should attempt for yourself to formalize this inductive process. The end result is a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in X \setminus (B(x_0, \varepsilon) \cup \cdots \cup B(x_{n-1}, \varepsilon))$ for all $n$. Since $X$ is sequentially compact, this sequence has a convergent subsequence, so let $a$ be a subsequential limit. Consider the open ball $B(a, \varepsilon/2)$. Then there exists some $m$ such that $x_m \in B(a, \varepsilon/2)$, and there exists some $n \geq m + 1$ such that $x_n \in B(a, \varepsilon/2)$. We have used the lemma we proved last time about subsequential limits twice here. Then

$$d(x_m, x_n) = d(x_m, a) + d(a, x_n) \leq \varepsilon,$$

which means that $x_n \in B(x_m, \varepsilon)$, even though we specifically chose

$$x_n \in X \setminus (B(x_0, \varepsilon) \cup \cdots \cup B(x_{n-1}, \varepsilon)) \subseteq X \setminus B(x_m, \varepsilon).$$

This gives us the contradiction we were seeking, so we can conclude that $X$ is totally bounded.

Finally, we have (c) implies (a), which is the hardest part of the proof of this theorem. Suppose that $X$ is complete and totally bounded and suppose for a contradiction that $\mathcal{U}$ is an open cover of $X$ which does not have a finite subcover. Since $X$ is totally bounded, by problem 5 we can cover $X$ with finitely many subsets of diameter at most 1. Since $\mathcal{U}$ has no finite subcover, it must be that one of these subsets cannot be covered by finitely many elements of $\mathcal{U}$. Let $A_0$ be such a set. Now since $X$ is totally bounded, we can apply problem 5 again to the set $A_0$ to cover it with finitely many subsets of diameter at most $1/2$. Proceeding in this way, we obtain a nested sequence of subsets

$$A_0 \supseteq A_1 \supseteq \cdots$$

such that $\text{diam}(A_n) \leq 1/(n + 1)$ and $A_n$ cannot be covered by finitely many elements of $\mathcal{U}$.

The fact that $A_n$ cannot be covered by finitely many elements of $\mathcal{U}$ implies in particular that $A_n$ is nonempty, so pick some point $x_n \in A_n$ and consider the sequence $(x_n)_{n \in \mathbb{N}}$. This sequence is Cauchy. Indeed, for any $\varepsilon \geq 0$, there exists some $n$ such that $1/(N + 1) \leq \varepsilon$, and then for any $m, n \geq N$, we see that $x_m \in A_m \subseteq A_N$ and $x_n \in A_n \subseteq A_N$, so

$$d(x_m, x_n) \leq \text{diam}(A_N) \leq 1/(N + 1) \leq \varepsilon.$$

Since $X$ is complete, this sequence must be convergent. Let $a = \lim x_n$.

Since $\mathcal{U}$ is an open cover, there exists some $U \in \mathcal{U}$ such that $a \in U$, and some open ball $B(a, r) \subseteq U$. Then there exists some $N$ such that $x_n \in B(a, r/2)$ for all $n \geq N$. Furthermore, there exists some $N'$ such that $1/(N' + 1) \leq r/2$. Then for any $n \geq \max\{N, N'\}$, we have $A_n \subseteq B(a, r)$. Indeed, for any $y \in A_n$, we have

$$d(a, y) \leq d(a, x_n) + d(x_n, y) \leq r/2 + \text{diam}(A_n) \leq r/2 + 1/(n + 1) \leq r/2 + 1/(N + 1) \leq r.$$
Thus we have $A_n \subseteq U$ as well, which is a contradiction since we specifically chose $A_n$ to not be coverable by finitely many elements of $U$.

2 Sequences in $\mathbb{R}^n$

A sequence $(x_n)_{n \in \mathbb{N}}$ is bounded if the set $E := \{x_n : n \in \mathbb{N}\}$ is bounded. We proved last time that every Cauchy sequence is bounded. Moreover, it is also clear that any subsequence of a bounded sequence is again bounded. Of course, in general, bounded sequences need not be convergent. In general it need not even be that bounded sequences have convergent subsequences: see problem 6. But this latter phenomenon cannot happen in $\mathbb{R}^n$.

**Theorem 2.1** (Bolzano-Weierstrass). If a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^n$ is bounded, it has a convergent subsequence.

*Proof.* Since the set $E := \{x_n : n \in \mathbb{N}\}$ is bounded, the closure $\bar{E}$ is as well, as we proved last time, which implies that it is compact by the Heine-Borel theorem. Then $(x_n)_{n \in \mathbb{N}}$ is a sequence in the compact set $\bar{E}$. Then $\bar{E}$ is sequentially compact by theorem 1.1, so it has a convergent subsequence.

**Corollary 2.2.** $\mathbb{R}^n$ is complete.

*Proof.* Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}^n$. Then it is bounded, so the Bolzano-Weierstrass theorem 2.1 guarantees that it has a convergent subsequence. But, as we know, Cauchy sequences with convergent subsequences must themselves be convergent.

3 Sample Problems

**Problem 1.** Give an example of a sequence in $\mathbb{R}$ with no convergent subsequence.

**Problem 2.** Give an example of a sequence in the open interval $(0, 1)$ with no convergent subsequence.

**Problem 3.** We know that the open interval $(0, 1)$ is not compact. Thus, by the theorem, it must either not be complete or not be totally bounded, or both. Is it complete? Is it totally bounded?

**Problem 4.** Let $X$ be a metric space.

(a) Show that, if $X$ is totally bounded, then it is bounded.

(b) Give an example to show that $X$ can be bounded without being totally bounded.

*Hint.* For (b), consider an infinite set with the discrete metric.

**Problem 5.** Let $X$ be a totally bounded metric space and let $E$ be a subset of $X$. Then, for any $\varepsilon \geq 0$, there exist finitely many sets of diameter at most $\varepsilon$ whose union equals $E$. 

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Proof. Since $X$ is totally bounded, there exist finitely many open balls $B(x_1, \varepsilon/2), \ldots, B(x_n, \varepsilon/2)$ whose union equals $X$. Moreover, we know that $\text{diam} B(x_i, \varepsilon/2) \leq \varepsilon$. Then $E_i := E \cap B(x_i, \varepsilon/2)$ are subsets of $X$ of diameter at most $\varepsilon$, and clearly $E = E_1 \cup \cdots \cup E_n$.

Problem 6. Show that it is possible to have a bounded sequence in a metric space which has no convergent subsequences.

Hint. Let $X$ be an infinite set with a discrete metric. Explain why the only convergent sequences in $X$ are the ones which are eventually constant. Then choose some sequence in $X$ all of whose points are distinct. Explain why this sequence must be bounded, and then explain why it cannot have any convergent subsequences.