Final Exam

Note. You must prove all assertions you make using only definitions and theorems we've discussed in class (or that are discussed in either Rudin or Ross, but you won't *need* to use anything that is discussed in one of these texts but wasn't discussed in class). The only exception to this rule is that you may use facts you remember about the standard functions of calculus (such as trigonometric functions and exponential functions).

Problem 1 (1 point each)**.** Determine whether each of the following statements is true or false.

(A) There exists a continuous function which maps $[0,1]$ onto $\mathbb{R}$.

(B) There exists a continuous function which maps $(0,1]$ onto $\mathbb{R}$.

(C) There exists a continuous function which maps $(0,1]$ onto the Cantor set.

(D) If a continuous function $f : [0,1) \to \mathbb{R}$ is uniformly continuous on $[1/2,1)$, then it is also uniformly continuous on $[0,1)$.

(E) There exists $a \in [0,1]$ such that $a \cdot 75^a = 37$.

(F) If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are bounded sequences in $\mathbb{R}$, then

$$\limsup_{n\to\infty}(x_n y_n) = \left(\limsup_{n\to\infty} x_n\right)\left(\limsup_{n\to\infty} y_n\right).$$

(G) If $\sum a_n$ is a convergent series in $\mathbb{R}$ and $(b_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$, then the series $\sum a_n b_n$ is also convergent.

(H) If $\sum a_n$ is an absolutely convergent series in $\mathbb{R}$ and $(b_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$, then the series $\sum a_n b_n$ is also absolutely convergent.

(I) The series $\displaystyle\sum_{n=0}^{\infty} \frac{n^{2015}}{2^n} x^n$ converges if and only if $x \in (-2,2)$.

(J) The series $\displaystyle\sum_{n=1}^{\infty} n^6 e^{-n^7}$ converges.

(K) If $f : [0,1] \to \mathbb{R}$ is a bounded function and $|f|$ is integrable, then f is also integrable.

(L) If $(f_n)_{n\in\mathbb{N}}$ is a uniformly convergent sequence of nonnegative continuous functions $[a,b] \to \mathbb{R}$ such that $\displaystyle\int_a^b f_n \le 2^{-n}$ for all $n \in \mathbb{N}$, and if $f := \lim f_n$, then $f(x) = 0$ for all $x \in [a,b]$.

(M) The union of two connected subsets of a metric space X is again connected.

(N) The intersection of two connected subsets of a metric space X is again connected.

(O) In any metric space X, any open ball $B(a,r)$ is connected.

Problem 2 (2 points)**.** Fix a nonempty subset E of a metric space X and then define $f : X \to \mathbb{R}$ by

$$f(a) = \inf\{d(a,x) : x \in E\}.$$

Show that f is uniformly continuous.

Problem 3. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $\lim\limits_{n\to\infty} |x_{n+1} - x_n| = 0$.

(a) (1-2 points) For 1 point, give an example where the sequence $(x_n)_{n\in\mathbb{N}}$ is not Cauchy. For 2 points, give an example where the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded but not Cauchy.

(b) (3 points) Let E be the set of subsequential limits of $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Show that E is connected. *Hint.* You might use problem 2 of problem set 4.

Problem 4 (3 points)**.** Show that, for any positive real number a, there exists a strictly increasing sequence $n_0 \lneq n_1 \lneq n_2 \lneq \cdots$ of positive integers such that

$$\sum_{k=0}^{\infty} \frac{1}{n_k} = a.$$

Problem 5 (3 points)**.** Let X be the set of all functions $\mathbb{N} \to \{0,1\}$. Define a metric d on X as follows. Let $d(x,x) = 0$ for any $x \in X$. Given two distinct elements $x, y \in X$, find the smallest nonnegative integer n such that $x(n) \neq y(n)$, and then set

$$d(x,y) := \frac{1}{2^n}.$$

You should spend a few minutes convincing yourself that d is a metric on X, but you need not submit a proof of this. I find it helpful to imagine the elements $x \in X$ as being the leaves on an infinite binary tree, where the value $x(n)$ specifies which of the two branches of the tree to follow at the nth stage; if you're struggling to make sense of this picture, ask me about it sometime.

Prove that X is compact.

Problem 6 (5 points)**.** Let $\mathcal{K}$ be the set of all compact intervals in $\mathbb{R}$, regarded as a metric space by restricting the Hausdorff metric (the one we defined in problem 8 of problem set 2). Let $\mathcal{R}$ be the set of bounded functions $f : \mathbb{R} \to \mathbb{R}$ such that f is integrable on every $[a,b] \in \mathcal{K}$, regarded as a metric space by restricting the supremum metric. Regard the product $\mathcal{K} \times \mathcal{R}$ as a metric space with the product metric (the one we defined in problem 10 of problem set 3). Consider the function $I : \mathcal{K} \times \mathcal{R} \to \mathbb{R}$ defined by

$$I([a,b], f) = \int_a^b f.$$

Is this function I uniformly continuous?