Advice

I remember finding my first analysis course extremely difficult. Part of that was because it was my first serious math course, but I think even if it hadn't been my first, I would have still found it quite difficult. To some extent this is unavoidable. If, like me, you also find that you're struggling in this class, there is nothing I can say to you that will suddenly make the subject easy. It is hard, and you should expect it to be hard. But here are some things that might have made my life easier if I had known them when I took analysis.

The mathematical process. Thinking about mathematics and writing about mathematics are two very different processes. Thinking about mathematics usually results in a piece of scratch paper that has been scribbled all over. There may be doodles, lots of things that are crossed out, equations that jump from one part of the page to entirely another, enormous logical leaps or even logic running exactly backwards... The point is, it would be almost useless to read the results of thinking about mathematics to anyone except the person who originally did the thinking.

Writing about mathematics is then the second stage. In this case, the mathematician looks at her or his scribbles and distills them into a single logical sequence. The result is a proof, the kind that you find in textbooks. They're usually arranged in such a way that it's almost like a computer could read through them and verify that each step follows logically from the last. It also often seems like something magical happens at some point in the proof that makes everything work: a number pulled out of a magic hat, a contradiction that catches you totally unawares...

This is useful to know for two reasons. The first is that it gives you a way of approaching solving new problems on your own. Make sure you have a thinking phase—don't jump right into the writing phase. If the first thing you do is try to produce a proof like the ones you see in textbooks, you might succeed for some easier problems, but you probably won't for more challenging problems. The second is that it tells you that reading a proof from top to bottom once through probably won't be good enough. The mathematician who wrote the proof didn't write it from top to bottom once through, so you're unlikely to understand it by reading it that way. You can certainly verify that one step follows logically from the last by doing this, but this is different from understanding the proof. There's another thinking phase that needs to happen when you're reading a proof.

Thinking when reading proofs. If you feel like something magical happens in a proof, or after you read it and verify its logic you still feel like you could never have come up with that proof on your own, you probably don't really understand the proof. There are several things you can do at this point to help you understand what's going on.

- One thing that is sometimes helpful is to consider an example. Often theorems are stated in a great deal of generality: maybe they're about arbitrary metric spaces, or maybe they're about an arbitrary Cauchy sequence... When you've just learned the definition of these concepts, this can feel overwhelming: you barely understand what the definition says, and now you need to be able to reason abstractly about the definition! What helps in this scenario is to produce an example. If the theorem is about a metric space X, take a metric space that you know and understand: maybe $\mathbb{R}^2$. Then follow along with the proof step by step. If the proof says "let p be a point in X," draw a dot in the middle of a plane on a piece of scratch paper. If the proof says "let N be an open ball of radius r around p," draw a circle around that dot that you drew. For every step in the proof, make sure you understand what's going on in the context of some specific example.

- Another thing that is useful is to produce a counterexample. Theorems often have some hypotheses. For example, maybe you come across the theorem that "in a compact metric space, every sequence has a convergent subsequence." Most likely, the mathematician who write this theorem didn't assume that the metric space was compact for no reason. When you see this theorem, you might think of an example of a non-compact metric space, like $\mathbb{R}$, and then show that you can actually find an example of a sequence where no subsequence is convergent. This will help you understand why compactness is a hypothesis in the theorem.

- Another thing you can do is to write down the statement of the theorem on a sheet of paper and then close your book. Try to prove the theorem yourself, and really try to get as far as you can. Maybe you'll find that you've made progress but there's one step, one statement that you really think should be true but you're having trouble proving it. When you really get stuck, then you open up your book again and look at the author's proof. Look at how they got over the hurdle you found yourself stuck on. When you haven't tried to write down a proof yourself, it may not be apparent why the author is doing all the things that he or she is doing. You may not even recognize where the hard parts of a proof are. After you've given it a sincere attempt, though, you're more likely to see where the hard parts of the proof are, and also to really understand and remember how to get over those hurdles.

You'll notice that doing all of these things is rather time-consuming, especially if you have to do it for every proof you read. But there really isn't any solution to that: really understanding things takes time. Do your best to understand as many proofs as you read

Thinking about new problems. Unfortunately, there is very little general advice I can give here. Every problem is different and no general approach will apply for all, or even many, of them. In any case, here are some thoughts.

- One thing I want to emphasize is doodling. Make sure you doodle. I remember many times during my first analysis class when I got so bogged down in epsilons and deltas, or trying to apply the triangle inequality in different random sequences, and failing for hours. Then I got tired and lied down in bed frustrated. Because I didn't have paper in front of me, I had to think about the problem in pictures instead of symbols, and suddenly the solution seemed obvious. When I went back to a piece of paper with that picture in my head, I knew exactly how I needed to manipulate the epsilons and deltas to get what I needed. Eventually I learned that it was much more efficient to skip all of that frustration and just doodle on some paper to begin with.

- Another thing that is really important is reading and understanding a lot of proofs. Once you've internalized the techniques used in proofs you find in textbooks, you can apply them to problems you're faced with. Also, reading proofs will give you a sense of what kind of writing is expected of you in a proof, which is very important if you haven't taken a proofs-based math class before.