Worksheet W3Wed: Laurent Series, Review

Problem 1. Find the Laurent series for the indicated function on the indicated region. What is c_{-1}?

(a) e^z/z^2, $0 < |z| < \infty$

(b) $1/(z^2 + z)$, $0 < |z| < 1$

(c) $z/(z+1)$, $0 < |z| < 1$

(d) $z/(z+1)$, $1 < |z| < \infty$

(e) $1/(z(z-1)(z-2))$, $0 < |z| < 1$

(f) $1/(z(z-1)(z-2))$, $1 < |z| < 2$

Problem 2. Let $f(z) = \dfrac{1}{(z-1)(z+1)}$.

(a) Find *two* Laurent series for $f(z)$ centered at $z = 1$. For each one, specify the annulus in which it converges.

(b) Integrate f over $C[1, 1]$.

Problem 3. Let $f(z) = \dfrac{1}{z(z-2)^2}$.

(a) Find *two* Laurent series for $f(z)$ centered at $z = 2$. For each one, specify the annulus in which it converges.

(b) Integrate f over $C[2, 1]$.

Problem 4. For any $n \geqslant 0$, calculate $\displaystyle\int_{C[0,1]} z^n \exp(1/z)\, dz$.

Problem 5. Calculate $\displaystyle\int_{C[0,1/2]} \frac{\exp(1/z)\, dz}{1-z}$.

Problem 6. Suppose f is holomorphic at w and fix a positive real number θ. For any small positive real number r, let $\gamma_r(t) = w + re^{it}$ for $t \in [0, \theta]$ be the arc of angle θ of the circle of radius r centered at w. Show that

$$i\theta f(w) = \lim_{r \to 0^+} \int_{\gamma_r} \frac{f(z)\, dz}{z - w}.$$

Possible hint. Use a strategy similar to the proof of Cauchy's integral formula (theorem 4.24 in BMPS). You'll need to first generalize exercise 4.4 and show that

$$\int_{\gamma_r} \frac{dz}{z - w} = i\theta$$

for any r.

Problem 7. Here is another strategy for calculating a real integral using complex analysis. The integral we'll evaluate is

$$\int_0^\infty \frac{\sin x\, dx}{x}.$$

(a) Convince yourself that no method you know from calculus will help you compute an antiderivative for $\sin(x)/x$.

(b) Suppose r is a small positive real number and R is a big one. Consider contour $C_{r,R}$ given by the horizontal segment $[-R, -r]$, followed by the *clockwise* semicircle $-\gamma_r$ from $-\epsilon$ through ir over to r, followed by the horizontal segment $[r, R]$, and finally the counterclockwise semicircle γ_R through iR back to $-R$. Integrate $f(z) := \exp(iz)/z$ over C.

(c) Show that $\displaystyle\lim_{R \to \infty} \int_{\gamma_R} f(z)\, dz = 0$.

(d) Use problem 6 to show that $\displaystyle\lim_{r \to 0^+} \int_{\gamma_r} f(z)\, dz = \pi i$.

(e) Put everything together to calculate $\displaystyle\int_0^\infty \frac{\sin x\, dx}{x}$.