Worksheet W3Tue: Analyticity, Multiplicities of Zeroes, Maximum Modulus Principle

Problem 1. Calculate a power series representation for $\operatorname{Log} z$ around $z = 1$.

Problem 2. Find the first 4 terms of the power series representation of $\exp(z)/(1-z)$ in a neighborhood of $z = 0$. Do this in two ways:

(a) Compute derivatives repeatedly and use corollary 8.5.

(b) Multiply the power series representation of $\exp(z)$ with that of $1/(1-z)$.

Problem 3. Find all zeroes *and* their multiplicities for the given functions. *Note for (f).* Define the function at $z = 0$ "by continuity."

(a) $(z-1)^2(z-2)^2(z-3)^3$

(b) $(z^2+1)\exp(z)$

(c) $\exp(z) - 1$

(d) $\operatorname{Log}(z)$

(e) $(az+b)/(cz+d)$ where $ad - bc \neq 0$

(f) $(\exp(z)-1)/z$

Problem 4. For each of the following, find a value of $z \in \bar{D}[0,1]$ which maximizes the modulus. Then find one which minimizes the modulus.

(a) $|e^z|$

(b) $|z^2 - 2|$

(c) $|z^2 + 2z + 1|$

(d) $|z^2 + 3z - 1|$

Problem 5. Fix $c \in \mathbb{C}$ and $R > 0$. Suppose f is holomorphic on $\bar{D}[0,1]$ and that $f(C[0,1]) \subseteq D[c,R]$. Prove that $f(D[0,1]) \subseteq D[c,R]$.

Problem 6. Fixing $c \in D[0,1]$, define $f(z) = \dfrac{z-c}{1-\bar{c}z}$.

(a) Verify that $f(c) = 0$.

(b) Verify that f is holomorphic on $\bar{D}[0,1]$. *Note.* Observe that, once you check the "$ad - bc \neq 0$" condition, f is a Möbius transformation. Thus it is sufficient to show that the denominator does not vanish inside $\bar{D}[0,1]$.

(c) Show that f maps $D[0,1]$ bijectively onto itself. *Possible hint.* Use the maximum modulus theorem and show that $|z| = 1$ implies $|f(z)| = 1$.