Worksheet W3Thu: Singularities, Review

Problem 1. Classify the type of singularity at $z = 0$ for each of the following functions. If it is a pole, identify the order of the pole.

(a) e^z/z (b) $(e^z - 1)/z^2$ (c) $z/(e^z - 1)$ (d) $z/(e^z - 1)(e^z - 2)$

Problem 2. Prove that the image of any punctured disk $P[0, \epsilon] := D[0, \epsilon] \setminus \{0\}$ under $f(z) = \exp(1/z)$ is $\mathbb{C} \setminus \{0\}$.

Problem 3. Prove the following complex version of l'Hôpital's rule. Suppose f and g are holomorphic at a, that g has a zero of order m at a, and f has a zero of order *at least* m at a. Then

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f^{(m)}(a)}{g^{(m)}(a)}.$$

Possible hint. Taylor series.

Problem 4. True or false? If f is entire, and it is bounded when restricted to the real axis, then f is constant.

Problem 5. Calculate $\displaystyle\int_{C[0,1]} \frac{dz}{\exp(z) - 1}$.

Problem 6. Let $Q = \{z \in \mathbb{C} : \operatorname{Re} z, \operatorname{Im} z > 0\}$ be the first quadrant.

(a) Find a conformal map from $D[i, 1]$ onto Q. Justify.

(b) Find a conformal map from $D[i, 1] \setminus \{i\}$ onto Q. Justify.

Problem 7. Calculate the value of the 68th derivative of $\exp(z^2)$ at $z = 0$.