Worksheet W3Mon: Cauchy's Integral Formulas, Liouville's Theorem

Problem 1. Calculate $\int_\gamma \frac{dz}{z^2-1}$ for the following paths.

(a) $\gamma = C[1,1]$ oriented counterclockwise.

(b) $\gamma = C[-1,1]$ oriented clockwise.

Problem 2. Integrate the following functions over $C[0,3]$.

(a) $\frac{1}{z-1}$

(b) $\frac{1}{z^2-4}$

(c) $\frac{\exp(z)}{z^3}$

(d) $\frac{1}{(z+4)(z^2+1)}$

(e) $\frac{\exp(2z)}{(z-1)^2(z-2)}$

(f) $\operatorname{Log}(z-4i)$

Problem 3. Suppose f is entire and $u = \operatorname{Re} f$ is bounded. Show that f is constant. *Possible hint*. We haven't assumed that f itself is bounded, so we can't apply Liouville's theorem directly. But... Can we compose f with *something* to get a bounded entire function?

Problem 4. Suppose f is entire and there exists an open ball $D[a,\epsilon]$ such that $f(\mathbb{C})$ is disjoint from $D[a,\epsilon]$. Show that f is constant.

Problem 5. Suppose $f : D[0,1] \to D[0,1]$ is holomorphic. Show that

$$|f'(w)| \leqslant \frac{1}{1-|w|}$$

for all $w \in D[0,1]$. *Possible hint*. Use Cauchy's integral formula for the derivative.

Problem 6. Give two examples of regions in $\mathbb{C}$ on which $\exp(1/z)$ has an antiderivative. Both your examples should be as large as possible.

Problem 7. In this problem, you will evaluate the following *real* integral using complex analysis.

$$\int_{-\infty}^{\infty} \frac{x\sin(x)\,dx}{x^2+1}.$$

The calculation is similar in spirit, but a little harder, than example 5.14 in BMPS.

(a) Convince yourself that no method you know from calculus will help you compute an antiderivative for $x\sin(x)/(x^2+1)$. (Or is there...?)

(b) For a real number $R > 1$, let α_R be the straight line path from $-R$ to R and let β_R be the semicircle from R to $-R$. Let γ_R be the semicircular closed path given by α_R followed by β_R. Evaluate

$$\int_{\gamma_R} f(z)\,dz \quad \text{for} \quad f(z) = \frac{z\exp(iz)}{z^2+1}.$$

(c) Show that $\left|\int_{\beta_R} f(z)\,dz\right| \leqslant \frac{R^2}{R^2-1}\int_0^\pi e^{-R\sin t}\,dt.$

(d) Use (c) to show that $\lim_{R\to\infty} \int_{\beta_R} f(z)\,dz = 0.$

Possible hints. Use symmetry of sin to argue that the integral of $e^{-R\sin t}$ over $[0,\pi]$ is twice the integral of the same function over $[0,\pi/2]$. Then consider the fact that, on the interval $[0,\pi/2]$, the graph of sin stays above the line connecting its endpoints.

(e) Put (b) and (d) together to evaluate $\int_{-\infty}^{\infty} \frac{x\sin(x)\,dx}{x^2+1}$.