Worksheet W3Fri: Residues, Review

Problem 1. Let f be holomorphic at a. Show that

$$\operatorname*{Res}_{z=a} \frac{f(z)}{(z-a)^{n+1}} = \frac{f^{(n)}(a)}{n!}$$

for any integer $n \geqslant 0$. Conclude that Cauchy's theorem and Cauchy's integral formula(s) can be derived from the residue theorem.

Problem 2. Find the following residues.

(a) $\operatorname*{Res}_{z=i} \dfrac{z^2-1}{(z^2+1)^2}$.

(b) $\operatorname*{Res}_{z=0} \dfrac{1}{e^z-1}$

(c) $\operatorname*{Res}_{z=0} \dfrac{1+e^z}{z^4}$

(d) $\operatorname*{Res}_{z=0} \dfrac{\exp(z^2)}{(z-1)^2}$

(e) $\operatorname*{Res}_{z=0} \dfrac{1+e^z}{z^2} + \dfrac{1}{z}$

(f) $\operatorname*{Res}_{z=0} \dfrac{1}{z^3(z+4)}$

Problem 3. Calculate the following integrals.

(a) $\displaystyle\int_{C[0,1]} \frac{dz}{e^z-1}$

(b) $\displaystyle\int_{C[0,2]} \frac{dz}{(z+1)^3}$

(c) $\displaystyle\int_{C[0,1]} \frac{z\,dz}{z^2+2z+5}$

(d) $\displaystyle\int_{C[0,1/2]} \frac{e^z\,dz}{z(1-z)^3}$

(e) $\displaystyle\int_{C[0,2]} \frac{e^z\,dz}{z(1-z)^3}$

(f) $\displaystyle\int_{C[0,1]} \frac{e^z\,dz}{z^2}$.

Problem 4. Suppose f is entire and $a, b \in \mathbb{C}$ with $a \neq b$.

(a) For $R > |a|, |b|$, evaluate $\displaystyle\int_{C[0,R]} \frac{f(z)\,dz}{(z-a)(z-b)}$.

(b) Use (a) to prove Liouville's theorem, ie, that any bounded entire function must be constant. *Possible hint*. $R \to \infty$.

Problem 5. Suppose f is holomorphic with a zero of multiplicity m at $z = a$. Show that

$$\operatorname*{Res}_{z=a} \frac{f'(z)}{f(z)} = m.$$

Problem 6. If f and g have simple poles at $z = a$, show that fg has a pole of order 2 at a and find a formula for the residue of fg at a.