Worksheet W2Fri: Cauchy's Theorem

Problem 1. Evaluate $\displaystyle\int_{C[0,r]} \frac{dz}{z^2 - 2z - 8}$ for $r = 1, 3, 5$.

Problem 2. Evaluate $\displaystyle\int_{\gamma} \left(z - \frac{1}{z}\right) dz$ for γ the straight line path from 1 to i. Do this problem in as many ways as you can.

Problem 3. Evaluate $\displaystyle\int_{C[0,1/2]} (z^2+1)^{1/2}\, dz$.

Problem 4. Show that $\displaystyle\int_{C[0,1]} \frac{dz}{z^3(z^2+10)} = \int_{C[0,2]} \frac{dz}{z^3(z^2+10)}$.

Problem 5. Show that

$$\int_{C[0,2]} \frac{dz}{z^3+1} = 0.$$

Possible hints. Use Cauchy's theorem to argue that the integral over $C[0,2]$ is the same as that over $C[0,R]$ for any $R \geqslant 2$. Then show that the limit as $R \to \infty$ of the integrals over $C[0,R]$ tends to 0.

Problem 6. Suppose f is entire. Calculate $\displaystyle\int_0^{2\pi} f(1+3e^{i\theta})e^{2i\theta}\, d\theta$.

Problem 7. Suppose G is an open set containing $\bar{D}[0,1]$. Let $f : G \setminus \{0\} \to \mathbb{C}$ be a holomorphic function which is *bounded near 0* (ie, there exists $M > 0$ and $\epsilon > 0$ such that $|f(z)| \leqslant M$ for all $z \in D[0,\epsilon]$). Show that

$$\int_{C[0,1]} f = 0.$$