Worksheet W1Thu: Möbius Transformations

Problem 1. (a) Suppose $x_0, y_0, r \in \mathbb{R}$ with $r > 0$. Show that the equation of a circle $(x - x_0)^2 + (y - y_0)^2 = r^2$ can be re-written in the form
$$\alpha(x^2 + y^2) + \beta x + \gamma y + \delta = 0$$
for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\alpha \neq 0$, and $\beta^2 + \gamma^2 > 4\alpha\delta$.

(b) Conversely, suppose we have $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha \neq 0$ and $\beta^2 + \gamma^2 > 4\alpha\delta$. Show that the equation $\alpha(x^2 + y^2) + \beta x + \gamma y + \delta = 0$ can be re-written in the form
$$(x - x_0)^2 + (y - y_0)^2 = r^2$$
for $x_0, y_0, r \in \mathbb{R}$ and $r > 0$.

Problem 2. Show that the Möbius transformation
$$f(z) = \frac{1 + z}{1 - z}$$
maps the unit circle bijectively onto the imaginary axis. *Possible hints*. Find a formula for $f(e^{i\theta})$, and in particular for $\operatorname{Im} f(e^{i\theta})$, in terms of real-valued functions of θ that you're familiar with from calculus. Then use calculus to show that the function $\theta \mapsto \operatorname{Im} f(e^{i\theta})$ is a bijection from $(0, 2\pi)$ onto $\mathbb{R}$.

Problem 3. Prove that any Möbius transformation different from the identity has at most two fixed points. (A *fixed point* of a function f is a point z such that $f(z) = z$.)

Problem 4. For $|a| < 1$, recall from your comprehension check that the function
$$f_a(z) = \frac{z - a}{1 - \bar{a}z}$$
is a Möbius transformation with $f_a^{-1} = f_{-a}$. Show that f_a maps the unit disk $D[0, 1]$ bijectively onto itself. *Possible hints*. Show first that it maps the unit circle to itself. Then show that $f_a(0) \in D[0, 1]$, and use the fact that a continuous function must map connected sets to connected sets to show that $f_a(D[0, 1]) \subseteq D[0, 1]$. Finally, put all of this together to show that f_a maps $D[0, 1]$ bijectively onto itself.

Problem 5. Consider the Möbius transformation
$$f(z) = \frac{2z}{z + 2}.$$
Calculate $f(0)$, $f(\pm 2)$, $f(\infty)$, and $f(-1 - i)$. Then, without calculating any other values of f, draw a picture of the image under f of each of the following sets. Note that all of the "lines" below are assumed to contain ∞. *Hint*. Recall that f is conformal, ie, that it preserves angles.

(a) The real axis.

(b) The imaginary axis.

(c) The line where real and imaginary parts are equal.

(d) The circle of radius 2 centered at 0.

(e) The circle of radius 1 centered at 1.

(f) The circle of radius 1 centered at -1.

Problem 6. Write down at least two different cross-ratios $f(z) = [z, a, b, c]$ such that f maps the circle $C[1 + i, 1]$ onto the real axis plus ∞. Can you arrange it so that one of your cross ratios sends the inside of the circle to the upper half plane, and the other sends the inside to the lower half plane?

Problem 7. Let $f(z) = 1/z$ and fix $a \in \mathbb{R} \setminus \{0\}$. Show that f maps the horizontal line $\operatorname{Im}(z) = a$ to the circle centered at $-i/2a$ with radius $1/2|a|$. Do this in two ways:

(a) Using the parametrization of the line $\operatorname{Im}(z) = a$ given by $x + ai$ for $x \in \mathbb{R}$. *Hint*. If $f(x) = u + vi$ for $u, v \in \mathbb{R}$, show that $u^2 + (v + 1/(2a))^2 = 1/4a^2$.

(b) Using the parametrization of the line $\operatorname{Im}(z) = a$ given by $a\tan(\theta) + ai$ for $\theta \in (-\pi, \pi)$. *Hint*. Double-angle formulas for trigonometric functions might end up being useful.

Problem 8. Find a Möbius transformation that maps the closed unit disk to $\{x + iy : x + y \geqslant 0\}$.

Problem 9. Let S^2 be the unit sphere in $\mathbb{R}^3$ and $\phi : S^2 \setminus \{(0,0,1\} \to \mathbb{C}$ the map given by

$$\phi(x, y, z) = \frac{x + iy}{1 - z}.$$

In other words, ϕ is the restriction of stereographic projection to the sphere minus the north pole. Verify that the inverse of this map is given by

$$\phi^{-1}(p + iq) = \left(\frac{2p}{p^2 + q^2 + 1}, \frac{2q}{p^2 + q^2 + 1}, \frac{p^2 + q^2 - 1}{p^2 + q^2 + 1} \right)$$

for all $p + iq \in \mathbb{C}$.