Worksheet W1Fri: Sequences and Series

Problem 1. Calculate $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$. *Hint*. Use partial fractions to rewrite the summands, and then consider the partial sums of this series.

Problem 2. For each of the following power series, find a rational function that the power series represents inside its radius of convergence.

(a) $\sum_{k=1}^{\infty} k(z-1)^{k-1}$ (b) $\sum_{k=2}^{\infty} k(k-1)z^k$

Problem 3. Calculate the radius of convergence of the following:

(a) $\sum n!z^n$ (b) $\sum z^n/e^n$ (c) $\sum z^n/n$ (d) $\sum n^2 z^n$

Problem 4. Suppose $a_1, a_2, \cdots$ and a are all complex numbers. For each of the following statements, prove any implication that is true, and provide a counterexample for any implication that is false.

(a) $\lim a_n = 0$ if and only if $\lim |a_n| = 0$.

(b) $\lim a_n = a$ if and only if $\lim |a_n| = |a|$.

(c) If $\lim a_n = a$, then (a_n) is bounded (ie, that there exists an M such that $|a_n| \leqslant M$ for all n).

(d) If $\lim a_n = a$ and x is an *accumulation point* of (a_n) (ie, for every $\epsilon > 0$ and every N, there exists $n > N$ such that $|a_n - x| < \epsilon$), then $x = a$.

Problem 5. Determine whether or not $\sum_{k=1}^{\infty} \frac{2k}{k^2+1}$ converges.

Problem 6. (a) Suppose (f_n) is a sequence of functions $f_n : G \to \mathbb{C}$ which converges uniformly to 0. Show that, for any sequence (z_n) in G, we have $\lim f_n(z_n) = 0$.

(b) Let $f_n : D[0,1] \to \mathbb{C}$ be given by $f_n(z) = z^n$. Show that f_n converges pointwise, but not uniformly, to 0. *Hint*. To show the convergence is not uniform, apply part (a) with the sequence $z_n = e^{-1/n}$.

Problem 7. Let (c_k) be a sequence.

(a) If (c_k) is bounded, show that $\sum c_k z^k$ has radius of convergence at least 1.

(b) If $\lim c_k \neq 0$, show that $\sum c_k z^k$ has radius of convergence at most 1.