Worksheet 9: Divisibility Rules, Linear Congruences, Review

Problem 1. Prove that a number is divisible by 9 if and only if the sum of its digits is divisible by 9.

Problem 2. Find an inverse of the given integer modulo the given integer, if possible. If it is not possible, explain why not.

(a) 2 mod 31

(b) 7 mod 31

(c) 5 mod 27

(d) 3 mod 27

Problem 3. Suppose $x, y \in \{0, 1, \ldots, 9\}$ are digits such that $495 \mid 273x49y5$. Find x and y.

Problem 4. Solve the given congruence, if possible. If it is not possible, explain why not.

(a) $2x \equiv 3 \bmod 11$

(b) $5x \equiv 3 \bmod 15$

(c) $3x \equiv 2 \bmod 7$

(d) $5x \equiv 17 \bmod 101$

Problem 5. Let n be an integer whose decimal representation is $a_r \cdots a_1 a_0$, where $a_i \in \{0, 1, \ldots, 9\}$ for all i. Show that n is divisible by 11 if and only if the alternating sum of its digits $a_0 - a_1 + a_2 - \cdots + (-1)^r a_r$ is divisible by 11.

Problem 6. Show that no number whose digits add up to 15 can be a perfect square.

Problem 7. Let n be a positive integer and a any integer. Observe that the congruence

$$ax \equiv 0 \bmod n$$

always has $x \equiv 0$ as a solution. Prove that this congruence has a *unique* solution (ie, any two solutions are congruent modulo n) if and only if $\gcd(a, n) = 1$.

Problem 8. Let a and b be two positive integers. By the fundamental theorem of arithmetic, there exists a finite set of primes $p_1, \ldots, p_n$ and some integers $e_1, \ldots, e_n, f_1, \cdots, f_n \geqslant 0$ such that $a = p_1^{e_1} \cdots p_n^{e_n}$ and $b = p_1^{f_1} \cdots p_n^{f_n}$. Show that

$$\gcd(a, b) = p_1^{\min\{e_1, f_1\}} \cdots p_n^{\min\{e_n, f_n\}}$$

and that

$$\operatorname{lcm}(a, b) = p_1^{\max\{e_1, f_1\}} \cdots p_n^{\max\{e_n, f_n\}}.$$

Problem 9. How many integers n are there such that 12^{12} the least common multiple of 6^6, 8^8, and n? *Hint*. Look at prime factorizations.

Problem 10. For all integers $n \geqslant 0$, we define a sequence of integers a_n as follows. We have $a_0 = 0$ and $a_1 = 1$, and then, for all $n \geqslant 2$, the decimal representation of a_n is obtained by writing the digits of a_{n-1} followed by the digits of a_{n-2}. For example:

$$\begin{aligned} a_2 &= 10 \\ a_3 &= 101 \\ a_4 &= 10110 \\ &\vdots \end{aligned}$$

Prove that $11 \mid a_n$ if and only if $6 \mid n$.