Worksheet 8: Euclidean Algorithm, Enumerating Primes, Review

Problem 1. Compute the following gcds using the Euclidean algorithm. Then work backwards through the Euclidean algorithm to express this gcd as a linear combination of the two numbers.

(a) $\gcd(105, 27)$

(b) $\gcd(1479, 272)$

Problem 2. How many prime numbers are less than 121?

Problem 3. Show that $\gcd(21n + 4, 14n + 3) = 1$ for all positive integers n.

Problem 4. Prove that there exist no integers x and y such that $1691x + 1349y = 1$.

Problem 5. Prove that, for any integer $n \geqslant 2$ and any collection of sets $A_1, \dots, A_n$ inside some universal set, we have

$$\overline{A_1 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.$$

Problem 6. Suppose n is a positive integer. If a is an integer, an integer b is called an *inverse of* a *modulo* n if $ab \equiv 1 \bmod n$.

(a) Show that 3 has an inverse modulo 17.

(b) Show that 3 does *not* have an inverse modulo 18.

Problem 7. Prove that if $a \mid n$ and $b \mid n$ with $\gcd(a, b) = 1$, then $ab \mid n$ using...

(a) Bézout's theorem.

(b) Ste17, Lemma 1.1.17.

Problem 8. Do there exist integers x and y such that $172x + 20y = 1000$? Justify.

Problem 9. Suppose $\gcd(a, b) = 1$. Is it true that $\gcd(ab, a + b) = 1$? Justify.

Problem 10. Consider the real number

$$\alpha = 0.235711131719\cdots,$$

whose digits after the decimal point are obtained by stringing together the decimal representations of all of the primes. Show that α is irrational. *Hint*. Use contradiction. You may use the fact that a number whose decimal expansion never repeats is irrational (ie, the converse of a problem from a previous worksheet, which we have not proved yet). You might decide to use Dirichlet's theorem on arithmetic progressions (Ste17, theorem 1.2.7) by considering the progression $10^{n+1}x + 1$ for some appropriate choice of n.