Worksheet 7: Induction, Prime Factorization

Problem 1. Prove that

$$\frac{(2n)!}{2^n \cdot n!}$$

is an integer for all $n \geqslant 0$.

Problem 2. Prove that $15 \mid 2^{4n} - 1$ for all non-negative integers n.

Problem 3. Let $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geqslant 3$. Show that F_n is even if and only if $3 \mid n$.

Problem 4. Suppose $p_1, \ldots, p_n$ are distinct primes. Show that $\sqrt{p_1 \cdots p_n}$ is irrational.

Problem 5. Let $p_1, p_2, p_3, \ldots$ be a list of the primes in increasing order. Prove that $p_n \leqslant p_1 \cdots p_{n-1} - 1$ for all $n \geqslant 3$.

Problem 6. Let $a \geqslant 2$ be an integer. By the fundamental theorem of arithmetic, there exist distinct primes $p_1, \ldots, p_n$ and positive integers $e_1, \ldots, e_n \in \mathbb{N}$ such that $a = p_1^{e_1} \cdots p_n^{e_n}$. Show that a is a perfect square if and only if e_i is even for all $i = 1, \ldots, n$.