Worksheet 6: Disproofs, More Sets

Notation. For any integer k, let $k\mathbb{Z} = \{kx : x \in \mathbb{Z}\}$ denote the set of multiples of k.

Problem 1. Show that

$$a\mathbb{Z} \cap b\mathbb{Z} = \text{lcm}(a, b)\mathbb{Z}$$

for all integers a and b.

Problem 2. For any $a \in \mathbb{N}$, show that $a\mathbb{Z} = \{x \in \mathbb{Z} : \gcd(a, x) = a\}$.

Problem 3. Is it true that there exist prime numbers p and q such that $p - q = 97$? Justify your answer.

Problem 4. Is it true that every integer $n \geqslant 2$ can be written in the form $n = a^2 + p$ where $a \in \mathbb{Z}$, and p is either prime or 1? Justify your answer.

Problem 5. Is it true that $A - (B \cap C) = (A - B) \cup (A - C)$ for all sets A, B, C? Justify your answer.

Problem 6. Determine whether each of the following statements is true. Justify your answer.

(a) $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$.

(b) $\mathscr{P}(A) \cup \mathscr{P}(B) = \mathscr{P}(A \cup B)$.

Problem 7. Is it true that $\{12x + 25y : x, y \in \mathbb{Z}\} = \mathbb{Z}$? Justify your answer.

Problem 8. Let p be a prime. Consider the statement "$p^2 + 2$ is prime if and only if $p = 3$." Is this a true statement? Justify your answer.