Worksheet 4: Contradiction

Problem 1. Prove that $\sqrt{6}$ is irrational.

Problem 2. Prove that there exist no integers a and b such that $21a + 30b = 1$.

Problem 3. Suppose a and b are integers such that $a^2 + b^2 \equiv 0 \mod 4$. Show that a and b are not both odd.

Problem 4. Show that, if n is composite, then there exists a divisor k of n such that $1 < k \leqslant \sqrt{n}$.

Problem 5. Let $n \geqslant 2$ be an integer and let d be the smallest divisor of n which is larger than 1. Show that d must be prime.

Problem 6. Prove that the sum of a rational and an irrational is irrational.

Problem 7. If a and b are positive real numbers, show that $a + b \geqslant 2\sqrt{ab}$.

Problem 8. Suppose $x \in \mathbb{R}$ and $0 < x < 1$. Show that $\dfrac{1}{x(1-x)} \geqslant 4$.

Problem 9. If a, b, c are integers such that $a^2 + b^2 = c^2$, show that either a or b must be even.

Problem 10. Prove that there exist no rational numbers x and y such that $x^2 + y^2 = 3$.