

Worksheet 11: Wilson's Theorem, Chinese Remainder Theorem, Review

Problem 1. Pair off the numbers $1, 2, \dots, 16$ into pairs of inverses mod 17. *Note.* Two of these numbers don't really make "pairs"...

Problem 2. Find the remainder when $15!$ is divided by 17.

Problem 3. Find an integer having remainders 1, 2, 5, 5, when divided by 2, 3, 6, 12, respectively. *Note.* This problem is attributed to 6th century mathematician Yi Xing.

Problem 4. When eggs in a basket are removed 2, 3, 4, 5, 6 at a time, there remain 1, 2, 3, 4, 5 eggs, respectively. When they are taken out 7 at a time, none are left over. Find the smallest number of eggs that could have been in the basket. *Note.* This problem is attributed to 7th century mathematician Brahmagupta.

Problem 5. Find an integer having remainders 3, 11, 15 when divided by 10, 13, 17, respectively. *Note.* This problem is attributed to 15th century mathematician Regiomontanus.

Problem 6. Find three consecutive integers, each having a square factor.

Problem 7. Find the smallest positive integer a such that $2 \mid a, 3 \mid a + 1, 4 \mid a + 2, 5 \mid a + 3$ and $6 \mid a + 4$.

Problem 8. Without using a calculator or computer, find the last two digits of 1032^{1032} . *Hint.* Find the remainders modulo 4 and 25, and then use the Chinese Remainder Theorem to find the remainder modulo 100.

Problem 9. Show that there exist infinitely many primes p such that $p + 2$ is *not* prime. *Remark.* If you remove the word "not" from this statement, you would obtain the statement of the twin prime conjecture, which is a famous unsolved problem in number theory!

Problem 10. The cells in a jail are numbered from 1 to 100 and their doors are activated from a central button. Activation opens a closed door and closes an open door. The k th time the button is pressed, all doors that are multiples of k are activated. If all doors are initially closed and the button is pressed 100 times, which doors will be open at the end? *Suggestion.* Try going through this process by hand with 20 instead of 100 first to get a feeling for the problem.