Worksheet 9: Divisibility Rules, Linear Congruences, Review

Problem 1. Prove that a number is divisible by 9 if and only if the sum of its digits is divisible by 9.

Problem 2. Find an inverse of the given integer modulo the given integer, if possible. If it is not possible, explain why not.

(a) 2 mod 31

(b) 7 mod 31

(c) 5 mod 27

(d) 3 mod 27

Problem 3. Suppose $x, y \in \{0, 1, \ldots, 9\}$ are digits such that $495 \mid 273x49y5$. Find x and y.

Solution. Observe that $2 \cdot 495 = 990$. This means that $10^3 \equiv 10 \bmod 495$. Thus,

$$\begin{aligned} 273x49y5 &= 5 + 10y + 9 \cdot 10^2 + 4 \cdot 10^3 + 10^4 x + 3 \cdot 10^5 + 7 \cdot 10^6 + 2 \cdot 10^7 \\ &\equiv 5 + 10y + 9 \cdot 10^2 + 4 \cdot 10 + 10^2 x + 3 \cdot 10 + 7 \cdot 10^2 + 2 \cdot 10 \bmod 495 \\ &= 1595 + 100x + 10y \\ &\equiv 110 + 100x + 10y \bmod 495 \end{aligned}$$

Since $x, y \in \{0, 1, \ldots, 9\}$, the only way this can happen is if the sum is 990, ie, if $x = y = 8$.

Problem 4. Solve the given congruence, if possible. If it is not possible, explain why not.

(a) $2x \equiv 3 \bmod 11$

(b) $5x \equiv 3 \bmod 15$

(c) $3x \equiv 2 \bmod 7$

(d) $5x \equiv 17 \bmod 101$

Problem 5. Let n be an integer whose decimal representation is $a_r \cdots a_1 a_0$, where $a_i \in \{0, 1, \ldots, 9\}$ for all i. Show that n is divisible by 11 if and only if the alternating sum of its digits $a_0 - a_1 + a_2 - \cdots + (-1)^r a_r$ is divisible by 11.

Problem 6. Show that no number whose digits add up to 15 can be a perfect square.

Solution. Suppose n is a number whose digits add up to 15. This means that $n \equiv 15 \equiv 6 \bmod 9$. Now suppose also that n is a perfect square, ie, that $n = a^2$ for some integer a. Then:

$$\begin{aligned} a \equiv 0 &\implies n \equiv 0 \\ a \equiv 1 &\implies n \equiv 1 \\ a \equiv 2 &\implies n \equiv 4 \\ a \equiv 3 &\implies n \equiv 0 \\ a \equiv 4 &\implies n \equiv 7 \\ a \equiv 5 &\implies n \equiv 7 \\ a \equiv 6 &\implies n \equiv 0 \\ a \equiv 7 &\implies n \equiv 4 \\ a \equiv 8 &\implies n \equiv 1 \end{aligned}$$

We reach a contradiction in all cases.

Problem 7. Let n be a positive integer and a any integer. Observe that the congruence

$$ax \equiv 0 \bmod n$$

always has $x \equiv 0$ as a solution. Prove that this congruence has a *unique* solution (ie, any two solutions are congruent modulo n) if and only if $\gcd(a, n) = 1$.

Solution. Suppose $\gcd(a, n) = 1$ and that x_1 and x_2 are both solutions. Then

$$0 \equiv ax_1 - ax_2 = a(x_1 - x_2).$$

Since $\gcd(a, n) \equiv 1$, there exists an integer b such that $ab \equiv 1 \bmod n$. Multiplying the above congruence through by b shows that $x_1 - x_2 \equiv 0 \bmod n$, ie, that $x_1 \equiv x_2 \bmod n$.

Conversely, suppose $d = \gcd(a, n) \neq 1$. Consider $x = n/d$. Since $d \mid x$, this is an integer. Moreover, since $d \mid a$, we know that a/d is an integer as well. Thus

$$a \cdot (n/d) = (a/d) \cdot n$$

is divisible by n, ie, $x = n/d$ is a solution to the congruence. Since $1 < n/d < n$, we clearly do not have $n/d \equiv 0 \bmod n$, so the congruence does not have a unique solution.

Problem 8. Let a and b be two positive integers. By the fundamental theorem of arithmetic, there exists a finite set of primes $p_1, \ldots, p_n$ and some integers $e_1, \ldots, e_n, f_1, \cdots, f_n \geqslant 0$ such that $a = p_1^{e_1} \cdots p_n^{e_n}$ and $b = p_1^{f_1} \cdots p_n^{f_n}$. Show that

$$\gcd(a, b) = p_1^{\min\{e_1, f_1\}} \cdots p_n^{\min\{e_n, f_n\}}$$

and that

$$\operatorname{lcm}(a, b) = p_1^{\max\{e_1, f_1\}} \cdots p_n^{\max\{e_n, f_n\}}.$$

Solution. Let $d = \gcd(a, b)$. No prime that's different from all of $p_1, \ldots, p_n$ can divide d, since then d would not divide a or b. Thus the prime factorization of d must be of the form $p_1^{a_1} \cdots p_n^{a_n}$ for some $a_1, \ldots, a_n \geqslant 0$. Since $p_1^{\min\{e_1, f_1\}} \cdots p_n^{\min\{e_n, f_n\}}$ divides both a and b, we must have $a_i \geqslant \min\{e_i, f_i\}$ for all i. If $a_i > \min\{e_i, f_i\}$ for some i, then d will not divide either a or b (since p_i will show up with a higher exponent in d than it does in either a or b). Thus it must be that $a_i = \min\{e_i, f_i\}$ for all i.

The proofs for lcms is similar and is omitted. Alternatively, you can also use the fact that $ab = \operatorname{lcm}(a, b) \gcd(a, b)$.

Problem 9. How many integers n are there such that 12^{12} is the least common multiple of 6^6, 8^8, and n? *Hint.* Look at prime factorizations.

Solution. We have the following prime factorizations:

$$\begin{aligned} 12^{12} &= (2^2 \cdot 3)^1 2 = 2^{24} \cdot 3^{12} \\ 6^6 &= (2 \cdot 3)^6 = 2^6 \cdot 3^6 \\ 8^8 &= (2^3)^8 = 2^{24} \end{aligned}$$

It follows that

$$\operatorname{lcm}(6^6, 8^8) = 2^{24} \cdot 3^6.$$

If n had a prime factor other than 2 or 3, then the lcm of k with any other integer would also have the same prime factor, so 12^{12} could not be the lcm. Thus n must be of the form $n = 2^{e_1} 3^{e_2}$. Now

$$\operatorname{lcm}(n, 2^{24} \cdot 3^6) = 2^{\max\{e_1, 24\}} \cdot 3^{\max\{e_2, 6\}}$$

so in order for this to equal 12^{12}, we need for $e_1 \leqslant 24$ and $e_2 = 12$. There are 25 options for e_1, so there are 25 integers n.

Problem 10. For all integers $n \geqslant 0$, we define a sequence of integers a_n as follows. We have $a_0 = 0$ and $a_1 = 1$, and then, for all $n \geqslant 2$, the decimal representation of a_n is obtained by writing the digits of a_{n-1} followed by the digits of a_{n-2}. For example:

$$\begin{aligned} a_2 &= 10 \\ a_3 &= 101 \\ a_4 &= 10110 \\ &\vdots \end{aligned}$$

Prove that $11 \mid a_n$ if and only if $6 \mid n$.

Solution. For a non-negative integer n, let $L(n)$ denote the number of digits in a_n. Then $L(0) = L(1) = 1$ and

$$L(n) = L(n) + L(n-1).$$

We claim that $L(n)$ is even if and only if $n \equiv 2 \bmod 3$. The proof of this is very similar to the proof of problem 4 on worksheet 7 and is omitted.

Now let $A(n)$ be the alternating sum of the digits of a_n. We will show by strong induction that we have the following.

$$A(n) = \begin{cases} 0 & \text{if } n \equiv 0 \bmod 6 \\ 1 & \text{if } n \equiv 1 \bmod 6 \\ -1 & \text{if } n \equiv 2 \bmod 6 \\ 2 & \text{if } n \equiv 3 \bmod 6 \\ 1 & \text{if } n \equiv 4 \bmod 6 \\ 1 & \text{if } n \equiv 5 \bmod 6 \end{cases}$$

It is straightforward to verify this for $n = 0, \cdots 5$. For the inductive step, observe that the definition of a_n implies that

$$A(n) = A(n-2) + (-1)^{L(n-2)} A(n-1)$$

for all $n \geqslant 2$. We now proceed in cases.

Suppose $n \equiv 0 \bmod 6$. Then $n - 1 \equiv 5 \bmod 6$ and $n - 2 \equiv 4 \bmod 6$. This implies that $n - 2 \equiv 1 \bmod 3$, so $n - 2$ is odd. Thus

$$A(n) = A(n-2) + (-1)^{L(n-2)} A(n-1) = 1 + (-1) \cdot 1 = 0.$$

The other five cases are similar calculations and are therefore omitted (but you should do them).

It follows from problem 5 that $11 \mid a_n$ if and only if $6 \mid n$.