Worksheet 6: Disproofs, More Sets

Notation. For any integer k, let $k\mathbb{Z} = \{kx : x \in \mathbb{Z}\}$ denote the set of multiples of k.

Problem 1. Show that
$$a\mathbb{Z} \cap b\mathbb{Z} = \operatorname{lcm}(a, b)\mathbb{Z}$$
for all integers a and b.

Solution. Let $m = \operatorname{lcm}(a, b)$. Any element of $m\mathbb{Z}$ is of the form mx for some integer x. Since $a, b \mid m$, we also have $a, b \mid mx$, so $mx \in a\mathbb{Z}$ and $mx \in b\mathbb{Z}$, ie, $mx \in a\mathbb{Z} \cap b\mathbb{Z}$. This shows that
$$m\mathbb{Z} \subseteq a\mathbb{Z} \cap b\mathbb{Z}.$$
On the other hand, suppose an integer n is in $a\mathbb{Z} \cap b\mathbb{Z}$. This means it is a multiple of both a and b. Since $m = \operatorname{lcm}(a, b)$ is the least common multiple, we must have $m \mid n$ (cf. problem 8 on worksheet 3). This shows that $n \in m\mathbb{Z}$, ie, that
$$a\mathbb{Z} \cap b\mathbb{Z} \subseteq m\mathbb{Z}.$$
Since we have shown inclusion both ways, we are done.

Problem 2. For any $a \in \mathbb{N}$, show that $a\mathbb{Z} = \{x \in \mathbb{Z} : \gcd(a, x) = a\}$.

Solution. The problem, in other words, is to show that the following statements are equivalent for any integer x.

(i) $a \mid x$

(ii) $\gcd(a, x) = a$

For (i) implies (ii): Note that a is a common factor of a and x, and it must be the greatest common factor since any factor of a must be less than it. Conversely, for (ii) implies (i): Note that $a = \gcd(a, x) \mid x$.

Problem 3. Is it true that there exist prime numbers p and q such that $p - q = 97$? Justify your answer.

Solution. It is not true. Suppose there did exist such primes. Then $p - q = 97$ is odd, so p and q must have opposite parity. But there is only one even prime and it is also the smallest prime, so we must have $q = 2$. Then $p - 2 = 97$ so $p = 99$, but $99 = 9 \cdot 11$ is not prime.

Problem 4. Is it true that every integer $n \geqslant 2$ can be written in the form $n = a^2 + p$ where $a \in \mathbb{Z}$, and p is either prime or 1? Justify your answer.

Solution. It is not true. Consider $n = 25$. If $25 = a^2 + p$ for p either prime or 1, then $a^2 < 25$, so $a^2 \in \{0, 1, 4, 9, 16\}$. But then $p = 25 - a^2 \in \{25, 24, 21, 14, 9\}$ and none of those options are prime or 1.

Problem 5. Is it true that $A - (B \cap C) = (A - B) \cup (A - C)$ for all sets A, B, C? Justify your answer.

Solution. It is true. Suppose $x \in A - (B \cap C)$. Then $x \in A$ but $x \notin B \cap C$, which means that either $x \notin B$ or $x \notin C$. In the former case, we have $x \in A - B$, and in the latter case, we have $x \in A - C$. Thus, in either case, we have $x \in (A - B) \cup (A - C)$. This shows that
$$A - (B \cap C) \subseteq (A - B) \cup (A - C).$$
Next, suppose $x \in (A - B) \cup (A - C)$. Then either $x \in A - B$ or $x \in A - C$. In either case, we have $x \in A$. Moreover, we either have $x \notin B$ or $x \notin C$, so in either case, we have $x \notin B \cap C$. Thus $x \in A - (B \cap C)$, which shows that
$$(A - B) \cup (A - C) \subseteq A - (B \cap C).$$
Since we have shown both inclusions, these two sets are equal.

Problem 6. Determine whether each of the following statements is true. Justify your answer.

(a) $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$.

(b) $\mathscr{P}(A) \cup \mathscr{P}(B) = \mathscr{P}(A \cup B)$.

Solution. Statement (a) is true. Indeed, $S \in \mathscr{P}(A) \cap \mathscr{P}(B)$ iff S is a subset of both A and of B, iff every element of S is also an element of both A and of B, iff $S \subseteq A \cap B$, iff $S \in \mathscr{P}(A \cap B)$.

Statement (b) is false. For example, consider $A = \{1\}$ and $B = \{2\}$. Then $\{1, 2\}$ is an element of $\mathscr{P}(A \cup B)$ but it is not in either $\mathscr{P}(A)$ or $\mathscr{P}(B)$.

Problem 7. Is it true that $\{12x + 25y : x, y \in \mathbb{Z}\} = \mathbb{Z}$? Justify your answer.

Solution. It is true. Observe that $\gcd(12, 25) = 1$ and that $\{12a + 25b : a, b \in \mathbb{Z}\}$ is precisely the set of multiples of 1 by Bézout's theorem. Since every integer is a multiple of 1, this shows that $\{12a + 25b : a, b \in \mathbb{Z}\} = \mathbb{Z}$.

Problem 8. Let p be a prime. Consider the statement "$p^2 + 2$ is prime if and only if $p = 3$." Is this a true statement? Justify your answer.

Solution. It is true. First of all, note that if $p = 3$, then $p^2 + 2 = 11$ is in fact prime. Conversely, suppose $p \neq 3$. Then $p \equiv \pm 1 \bmod 3$, so $p^2 + 2 \equiv (\pm 1)^2 + 2 = 1 + 2 \equiv 0 \bmod 3$, so $p^2 + 2$ cannot be prime.