Worksheet 1: Sets

Problem 1. Each of the following describes a finite set using set-builder notation. List all of the elements of each of these sets.

(a) $\{3x+2 : x \in \mathbb{Z}, 0 \leqslant x \leqslant 5\}$

(b) $\{x \in \mathbb{R} : x^3 + 5x^2 = -6x\}$

(c) $\{x \in \mathbb{R} : \cos x = 1, |x| \leqslant 10\}$

(d) $\{6a + 2b : a, b \in \mathbb{Z}, |a|, |b| \leqslant 4\}$

Problem 2. Identify the pattern and describe each of the following sets using set-builder notation.

(a) $\{0, 4, 16, 36, 64, 100, \dots\}$

(b) $\{\dots, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27, \dots\}$

(c) $\{\dots, -\frac{3}{2}, -\frac{3}{4}, 0, \frac{3}{4}, \frac{3}{2}, \frac{9}{4}, 3, \frac{15}{4}, \frac{9}{2}, \dots\}$

(d) $\{4, 9, 25, 49, 121, 169, \dots\}$

Problem 3. Draw a picture of each of the following sets in the x-y plane.

(a) $\{(x, y) : x \in [0, 1], y \in [1, 2]\}$

(b) $\{(x, x^2) : x \in \mathbb{R}\}$

(c) $\{(x, y) : x, y \in \mathbb{R}, x^2 + y^2 \leqslant 1\}$

(d) $\{(x, x + y) : x \in \mathbb{R}, y \in \mathbb{Z}\}$

(e) $[1, 2] \times \{1, 2, 3\}$

(f) $\mathbb{N} \times \mathbb{Z}$

Problem 4. Draw a picture of each of the following in $\mathbb{R}^3$.

(a) $[0, 1] \times [0, 1] \times [0, 1]$

(b) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\} \times [0, 1]$

Problem 5. Let $X = [-1, 3] \times [0, 2]$ and $Y = [0, 3] \times [1, 4]$ inside $\mathbb{R}^2$. Draw a picture of the following sets.

(a) $X \cup Y$

(b) $X \cap Y$

(c) $X - Y$

(d) $Y - X$

(e) $\overline{X}$

(f) $\overline{X} \cup Y$

Problem 6. Suppose A, B, and C are in a universal set U. Draw Venn diagrams for the following.

(a) $\overline{A \cup B}$

(b) $(B \cap C) - A$

(c) $(A - B) \cup (B \cap C)$

(d) $\overline{A \cap B \cap C}$

Problem 7. Draw a picture of each of the following sets inside $\mathbb{R}^2$.

(a) $\displaystyle\bigcup_{a \in \mathbb{R}} \{a\} \times [0, 1]$

(b) $\displaystyle\bigcap_{n \in \mathbb{N}} [-n, n] \times [-n, n]$

(c) $\displaystyle\bigcap_{n \in \mathbb{N}} \mathbb{Z} \times \{\dots, -2n, -n, 0, n, 2n, 3n, \dots\}$

(d) $\displaystyle\bigcup_{x \in [0,1]} [x, 1] \times [0, x^2]$

Problem 8. Let $S = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. Find all subsets of S.

Solution. Since S has 3 elements, it must have $2^3 = 8$ subsets. They are as follows. The first has cardinality 0, the next 3 have cardinality 1, the next 3 have cardinality 2, and the last has cardinality 3.

$$\emptyset \quad \{\emptyset\} \quad \{\{\emptyset\}\} \quad \{\{\emptyset, \{\emptyset\}\} \quad \{\emptyset, \{\emptyset\}\} \quad \{\emptyset, \{\emptyset, \{\emptyset\}\}\} \quad \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

Problem 9. Determine whether or not each of the following sets has a smallest element. If it does, say what the smallest element is.

(a) $\{4x + 6y \in \mathbb{N} : x, y \in \mathbb{Z}\}$

(b) $\{4x + 6y \in \mathbb{Z} : x, y \in \mathbb{Z}\}$

Solution. Observe that any element of either set must be even since $4x + 6y = 2 \cdot (2x + 3y)$. The first set contains 2 since $2 = 4 \cdot (-1) + 6 \cdot 1$, and 2 is the smallest positive integer, so this must be the smallest element.

The second set has no smallest positive integer. Roughly, the idea is that, as n varies, the expression

$$4 \cdot n + 6 \cdot (-n) = -2n$$

varies over all possible even integers, and there is no smallest even integer. This should formally be written as a proof by contradiction, which we'll talk about later in the course.

Problem 10. Some words can be used to describe words. For example, the word *verb* describes words like *breathe* and *grow*. The word *noun* describes words like *tree* and *absurdity*, and in fact, it also describes *verb* and *noun*. The word *pentasyllable* describes words like *imagination* and also *pentasyllable* itself. Let us say that a word is a *heteroseme* if it does not describe itself. For example, we've just seen that *verb* is a heteroseme while *noun* and *pentasyllable* are not. Is *heteroseme* a heteroseme?

Solution. Suppose *heteroseme* is not a heteroseme. That means it does describe itself. But by definition of *heteroseme*, that would mean that it does not describe itself. This is a contradiction. Conversely, suppose *heterosome* is a heteroseme. That means it does not describe itself. By definition of *heterosome*, this means that heteroseme must describe itself. This is again a contradiction. Thus *heterosome* is neither a heterosome nor not a heteroseme! This is the Grelling-Nelson paradox, which is a version of Russell's paradox.