Homomorphisms

A function $\varphi: G \to H$ is a <u>homomorphism</u> if it is operation-preserving, ie, $\varphi(ab) = \varphi(a)\varphi(b)$. ← might need to be rewritten if working with additive groups!

Here are a few of the important properties:

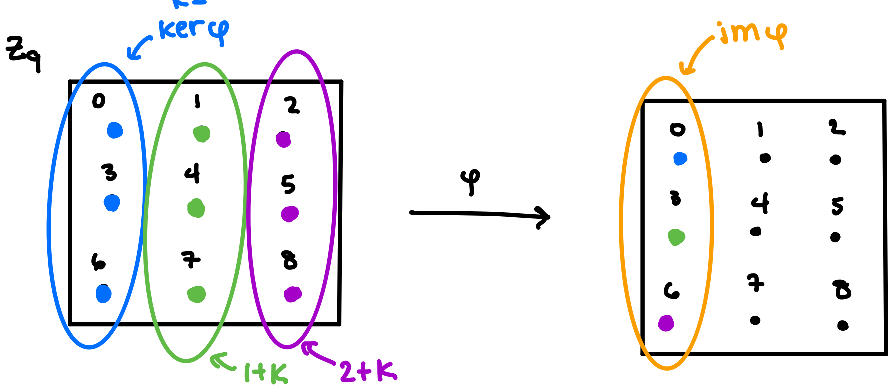
not necessarily normal!

- $\operatorname{im}\varphi = \{y \in H \mid \text{there exists } x \in G \text{ such that } \varphi(x) = y\}$ is a subgroup of H.
- $\ker\varphi = \{x \in G \mid \varphi(x) = e_H\}$ is a normal subgroup of G.

  identity of H

  allows us to form a factor group!
- Let $K = \ker\varphi$. Then $\varphi(a) = \varphi(b)$ iff $aK = bK$.
- In particular, φ is injective iff $K = \{e_G\}$ identity of G

Ex. $\varphi: \mathbb{Z}_9 \longrightarrow \mathbb{Z}_9$ given by $\varphi(x) = 3x$.



φ is a homomorphism:

$$\varphi(a+b) = 3(a+b) = 3a + 3b = \varphi(a) + \varphi(b)$$

since $\mathbb{Z}_9$ abelian

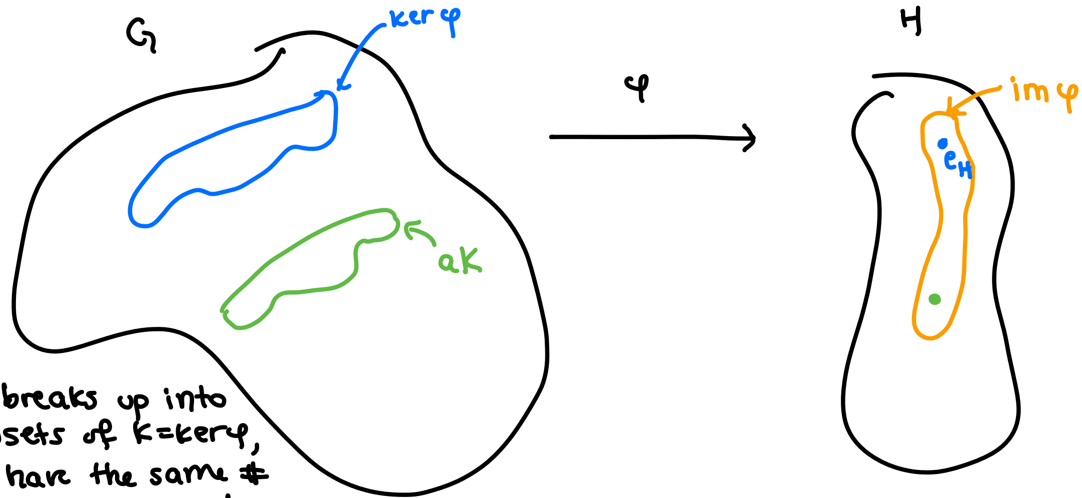
$\operatorname{im}\varphi = \{0,3,6\}$ is a subgroup of $\mathbb{Z}_9$

$\ker\varphi = \{0,3,6\}$ is a normal subgroup of $\mathbb{Z}_9$. Let $K = \ker\varphi = \{0,3,6\}$.

Let's think about the cosets of K.

$$K = 0+K = 3+K = 6+K = \{0,3,6\}$$
$$1+K = 4+K = 7+K = \{1,4,7\}$$
$$2+K = 5+K = 8+K = \{2,5,8\}$$

We can see that a pair of elements in domain have same image under φ iff they belong to the same coset of K.



G breaks up into cosets of $K = \ker\varphi$, all have the same # of elements, and everything in a coset gets mapped to same element of H.

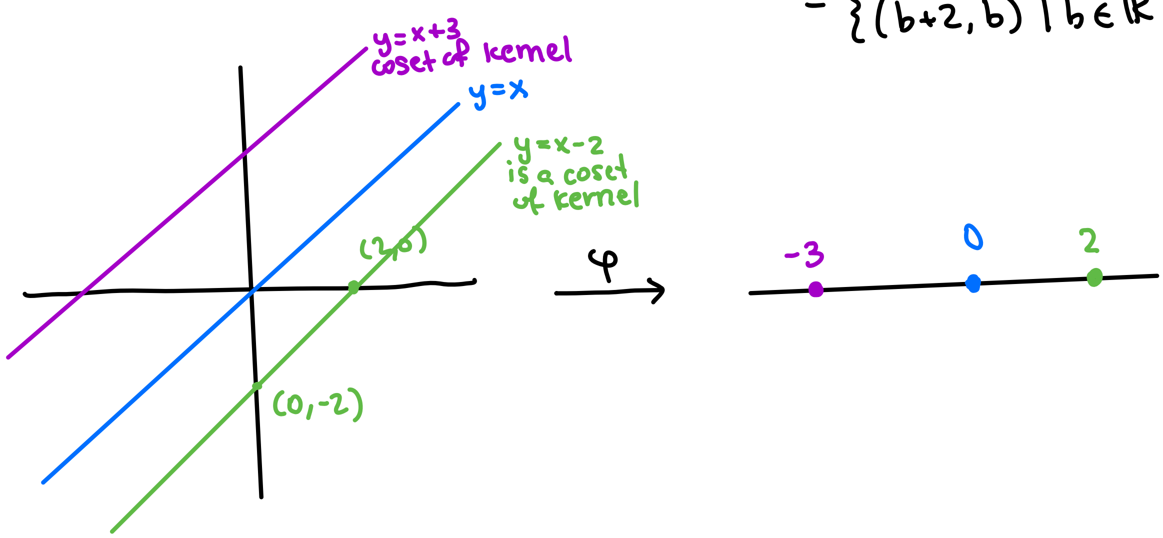
1. $\varphi : \underbrace{\mathbb{R}\oplus\mathbb{R}}_{\mathbb{R}^2} \longrightarrow \mathbb{R}$

$$\begin{aligned}\varphi(a,b)+\varphi(a',b') &= (a-b)+(a'-b') \\ &= (a+a')-(b+b') \\ &= \varphi(a+a',\, b+b') \\ &= \varphi((a+b)+(a'+b'))\end{aligned}$$

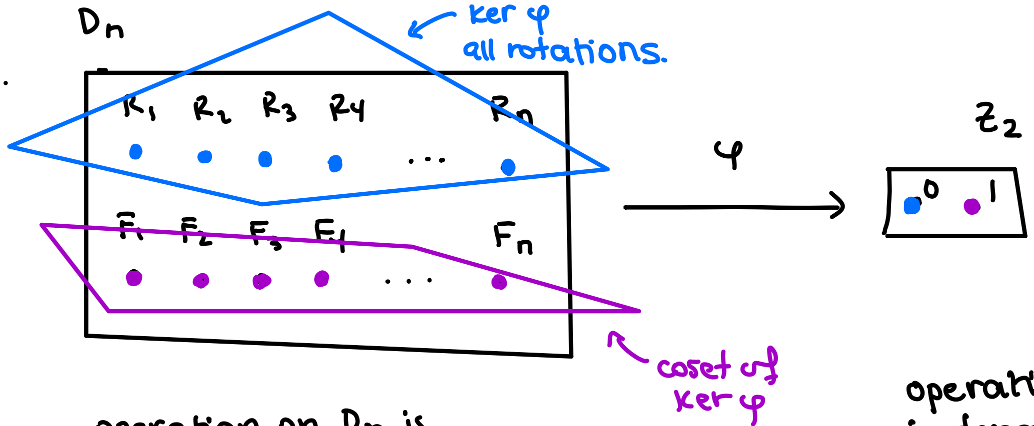
so φ is a homomorphism.

$$\begin{aligned}\ker\varphi &= \{(a,b) \mid \varphi(a,b)=0\} \\ &= \{(a,b) \mid a-b=0\} \\ &= \{(a,a) \mid a\in\mathbb{R}\}\end{aligned}$$

$$\begin{aligned}\varphi^{-1}(2) &= \{(a,b) \mid \varphi(a,b)=2\} \\ &= \{(a,b) \mid a-b=2\} \\ &= \{(a,b) \mid a=b+2\} \\ &= \{(b+2,b) \mid b\in\mathbb{R}\}\end{aligned}$$



2.



operation on D_n is denoted multiplicatively

operation on $\mathbb{Z}_2$ is denoted additively

So we want to show that $\varphi(ab) = \varphi(a) + \varphi(b)$ for all $a, b \in D_n$.

Case 1. a & b are both rotations

$\varphi(ab) = 0$ since ab is also a rotation

$\varphi(a) + \varphi(b) = 0 + 0 = 0$

Case 2: a rotation, b reflection.

$\varphi(ab) = 1$ since ab reflection.

$\varphi(a) + \varphi(b) = 0 + 1 = 1$

Case 3. a reflection, b rotation.
Case 4. a & b both reflections

} finish there!

---

16 elements.

9.25. $G = U(32)$ $H = \{1, 15\}$ G/H 8 elements, so by Lagrange, any elt has order 1, 2, 4, or 8.

$G = \{1, 3, 5, 7, 9, \ldots, 31\}$

H identity elt of G/H, so has order 1.

$(3H)^2 = 9H$

$(3H)^4 = (9H)^2 = 81H = 17H$

$(3H)^8 = (17H)^2 = 1H = H.$

- 3H is an elt of G/H of order 8, so G/H must be isomorphic to $\mathbb{Z}_8$ since $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ & $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ don't have any elements of order 8.

- $G/H = \langle 3H \rangle$, so $G/H \approx \mathbb{Z}_8$

8 elements $|\langle 3H \rangle| = |3H| = 8$ any cyclic group of order n is isomorphic to $\mathbb{Z}_n$.

$3H = 13H.$
$5H$
$19H$
$21H.$