Factor Groups

Recall: a subgroup H of G is normal if either of the following is satisfied:

- $aH = Ha$ for any $a \in G$
- $aHa^{-1} \subseteq H$ for any $a \in G$.

If H is normal, let G/H be the set of left cosets of H in G. This is a group under the operation $(aH)(bH) = abH$. This is well-defined because H is normal.

Ex. $G = \mathbb{Z}_9 = \{0,1,2,\ldots,8\}$ abelian group

$H = \langle 3 \rangle = \{0,3,6\}$ normal b/c G is abelian.

can form G/H

As a set, consists of cosets of H in G.

$$H = \{0,3,6\} = 3+H = 6+H$$
$$1+H = \{1,4,7\} = 4+H = 7+H$$
$$2+H = \{2,5,8\} = 5+H = 8+H$$

4+H is a coset of H. It contains 4 since H contains 0. Also, we just saw that 1+H contains 4. So, since 4 is in both 1+H & 4+H, we must have 4+H = 1+H.

$4+H = \{0+4, 3+4, 6+4\}$
$= \{4,7,1\} = 1+H.$

These are all of the cosets: $|G/H| = 3$.

$$G/H = \{H,\ 1+H,\ 2+H\}$$

elements of G/H are themselves sets!

There's a group operation:

$$(1+H) + (2+H) = (1+2)+H = 3+H = H$$
$$(1+H) + H = (1+0)+H = 1+H$$
$$(2+H) + H = 2+H$$

- H is the identity element of G/H.
- $2+H$ is the inverse of $1+H$, ie, $2+H = -(1+H)$.
- The inverse of 1 in $\mathbb{Z}_9$ is 8, and $8+H = 2+H$. It is also true that $8+H$ is the inverse of $1+H$ in G/H.
- $2+H = 8+H$

$$(1+H)+(2+H) \overset{?}{=} (1+H)+(8+H)$$
$$(1+H)+(2+H) = (1+2)+H = H, \qquad (1+H)+(8+H) = (1+8)+H = H$$

- The normality of H is crucial in ensuring we get the same answer.

Ex. Inside $D_3 = \{R_0, R_{120}, R_{240}, F_1, F_2, F_3\}$, we've seen that $H = \{R_0, R_{120}, R_{240}\}$ is a normal subgroup.

So we can form D_3/H.

$$H = \{R_0, R_{120}, R_{240}\} = R_{120}H = R_{240}H = R_0H$$
$$F_1H = \{F_1, F_2, F_3\} = F_2H = F_3H$$
$$D_3/H = \{H, F_1H\}$$

This is a group:

$$H\,(F_1H) = (R_0H)(F_1H) = (R_0F_1)H = F_1H$$
$$(F_1H)(F_1H) = F_1^2H = R_0H = H$$
$$(F_1H)(F_2H) = \underbrace{F_1F_2}_{\text{some rotation!}}H = H$$

- H is identity element.

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1. $\mathbb{Z}_{24} = \{0, 1, \ldots, 23\}$ — 24 elts

$\langle 8 \rangle = \{0, 8, 16\}$ — 3 elts

$\langle 8 \rangle$ has $\frac{24}{3} = 8$ cosets in $\mathbb{Z}_{24}$, ie, $\mathbb{Z}_{24}/\langle 8 \rangle$ is a group of order 8. Lagrange's thm tells us that order of any elt must divide 8.

$14 + \langle 8 \rangle = 6 + \langle 8 \rangle = \{6, 14, 22\}$

The order is the smallest k such that $k(6 + \langle 8 \rangle) = \langle 8 \rangle$

But $k(6+\langle 8 \rangle) = k6 + \langle 8 \rangle = \langle 8 \rangle$ iff $6k \in \langle 8 \rangle$.

comes from the definition of the group operation on G/H.

$k=1$ $\quad 6 \notin \langle 8 \rangle$

$k=2$ $\quad 12 \notin \langle 8 \rangle$.

[$k=3$ $\quad 18 \notin \langle 8 \rangle$] unnecessary, already know 3 can't be the answer

$k=4$ $\quad 24 = 0 \in \langle 8 \rangle$. ✓

So $6 + \langle 8 \rangle = 14 + \langle 8 \rangle$ has order 4.

2. $H = \langle 12 \rangle$ in $U(13)$. What is order of $4H$?

$H = \{1, 12\}$ $\qquad$ $12^2 = 144 \bmod 13 = 1$

Looking for smallest k such that $(4H)^k = H$, ie, $4^k H = H$, ie, $4^k \in H$.

4 $\qquad$ $4^2 = 16 = 3$ $\qquad$ $4^3 = 3 \cdot 4 = 12 \in H$

so 4 has order 3.

- D_{12} is isomorphic to a subgrp of S_n for some n. (Cayley).
  But don't know that $n=4$ (if $n=4$, would know $D_{12} \approx S_4$).
- D_{12} has a subgrp of index 2 (namely, rotations)
  S_4 has a subgrp of index 2 (namely, even permutations)
- Elements of order 2

  $\left\{ \begin{array}{l} (1\,2)(3\,4) \\ (1\,3)(2\,4) \\ (1\,4)(2\,3) \end{array} \right.$

  D_{12} — 13 elements (12 reflections, R_{180}).

  S_4 — 9 elements ($\binom{4}{2} = 6$ 2-cycles, 3 disjoint prods of 2-cycles)

  so not isomorphic!

no other elts of order 2 because:-
any elt of S_4 has disjoint cycle form
and order of elt is lcm of lengths of cycles
so only way the lcm can be 2 is if all cycles
in the disjoint cycle form are 2-cycles.
But we only have 4 numbers $\{1,2,3,4\}$, so
we can't have more than 2 disjoint 2-cycles!