Day 16 — Cayley's Theorem

Every group is isomorphic to a group of permutations.

Every finite group is isomorphic to a subgroup of S_n.

Ex. $U(5) = \{1,2,3,4\}$

This is isomorphic to a subgroup of S_4.

symmetric group, ie, all permutations on $\{1,2,3,4\}$, ie, all bijective functions $\{1,2,3,4\} \to \{1,2,3,4\}$

$$2 \in U(5) \rightsquigarrow T_2 : \begin{array}{l} 1 \longmapsto 2 \\ 2 \longmapsto 4 \\ 3 \longmapsto 1 \\ 4 \longmapsto 3 \end{array} = (1\ 2\ 4\ 3)$$

$$1 \in U(5) \rightsquigarrow T_1 : \begin{array}{l} 1 \longmapsto 1 \\ 2 \longmapsto 2 \\ 3 \longmapsto 3 \\ 4 \longmapsto 4 \end{array} = (1)$$

$$3 \in U(5) \rightsquigarrow T_3 : \begin{array}{l} 1 \longmapsto 3 \\ 2 \longmapsto 1 \\ 3 \longmapsto 4 \\ 4 \longmapsto 2 \end{array} = (1\ 3\ 4\ 2)$$

$$4 \in U(5) \rightsquigarrow T_4 : \begin{array}{l} 1 \longmapsto 4 \\ 2 \longmapsto 3 \\ 3 \longmapsto 2 \\ 4 \longmapsto 1 \end{array} = (1\ 4)(2\ 3)$$

$H = \{(1), (1\,2\,4\,3), (1\,3\,4\,2), (1\,4)(2\,3)\}$

is a subgroup of S_4, and the map

$$\begin{array}{l} U(5) \longrightarrow H \\ 1 \longmapsto (1) \\ 2 \longmapsto (1\,2\,4\,3) \\ 3 \longmapsto (1\,3\,4\,2) \\ 4 \longmapsto (1\,4\ 2\,3) \end{array}$$

Is an isomorphism of groups.

Rarely does anyone invoke the statement of Cayley's thm to prove something about groups. "Reducing" a statement about all groups to a statement about all permutation groups is not really much of a reduction.

But the thm is conceptually important: it's often very useful to think about group elements as permutations.

1. $$\underbrace{\underbrace{(1\ 4)}_{\text{2-cycle}}\ (1\ 3)\ (1\ 2)}_{3\ \text{2-cycles}}$$

Question doesn't specify 3-cycles must be disjoint. We do know that disjoint cycle notation is unique. So we do know that there's no way of doing this where 3-cycles are disjoint, but unclear if there might be a way where 3-cycles are not disjoint.

$(1\ 2\ 3\ 4)$ is a 4-cycle, so its odd.

Any 3-cycle is even, and the product of even permutations will again be even. So $(1\ 2\ 3\ 4)$ cannot be a product of 3-cycles.

$(1\ 2\ 3) = (1\ 3)(1\ 2)$ $\qquad$ $(4\ 7\ 5) = (4\ 5)(4\ 7)$

so $(1\ 2\ 3)$ is a product of 2 (ie, an even number of) transpositions (2-cycles).

2. Order of $(1\ 2\ 4\ 6\ 7\ 5)(3\ 8)$

The cycles are disjoint, so order is lcm of the lengths of the cycles. $\text{lcm}(6,2) = 6$.

3. Find subgrp of S_n isomorphic to $\mathbb{Z}_4$.

$\mathbb{Z}_4 = \{0,1,2,3\}$ operation addition mod 4.

$0 \rightsquigarrow T_0 :$ $0 \mapsto 0$, $1 \mapsto 1$, $2 \mapsto 2$, $3 \mapsto 3$. $\quad$ $2 \mapsto 2$, $1 \mapsto 1$, $3 \mapsto 3$, $4 \mapsto 4$ $= (1)$

$1 \rightsquigarrow T_1 :$ $0 \mapsto 1$, $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 0$ $\quad$ $2 \mapsto 1$, $1 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 2$ $= (1\ 3\ 4\ 2)$

$2 \rightsquigarrow T_2 :$ $0 \mapsto 2$, $1 \mapsto 3$, $2 \mapsto 0$, $3 \mapsto 1$ $\quad$ $2 \mapsto 3$, $1 \mapsto 4$, $3 \mapsto 2$, $4 \mapsto 1$ $= (1\ 4)(2\ 3)$

$3 \rightsquigarrow T_3 :$ $0 \mapsto 3$, $1 \mapsto 0$, $2 \mapsto 1$, $3 \mapsto 2$ $\quad$ (finish this) (*)

If I associate to each element of $\mathbb{Z}_4$ a number in $\{1,2,3,4\}$, I can regard each of these as a element of S_4.

$0 \rightsquigarrow 2$
$1 \rightsquigarrow 1$
$2 \rightsquigarrow 3$
$3 \rightsquigarrow 4$

completely arbitrary

$H = \{ (1), (1\,3\,4\,2), (14)(23), (*) \}$

will ~~tso~~ be isomorphic to $\mathbb{Z}_4$.

$(1\ 4\ 3\ 2)$ represents a function $\{1,2,3,4\} \to \{1,2,3,4\}$

$$1 \longmapsto 4$$
$$4 \longmapsto 3$$
$$3 \longmapsto 2$$
$$2 \longmapsto 1$$

sets don't care about order, and the domain & codomain of $(1\ 4\ 3\ 2)$ is the same. It's just elements are moving around.