I don't quite understand Exercise 3.1.5..

I don't know how to approach question 3.1.5 in the CC. Can you walk through how to prove the generalized Elimination Theorem?

I think 3.1.5 is really confusing to me.

Discussion. The proof for exercise 3.1.5 is basically identical to the proof of the elimination theorem. Once you understand the setup, it's just a matter of going through the proof of the elimination theorem given in the textbook and *almost* re-writing that proof word-for-word. Most of the difficulty is in the abstraction of the statement, so let me focus on a special case to make the statement more transparent.

Let's say we're working with three variables x, y, z and we want to eliminate x and y. In other words, we're in the polynomial ring $k[x, y, z]$ and looking at monomial orders of 2-elimination type.

First of all, some examples of monomial orders of 2-elimination type.

- The "easy" example is lex order where $x > y > z$. This is of 2-elimination type because, under lex order, any monomial which involves x or y is bigger than any monomial which involves only z.

  I say "easy" in quotes because, while this is a conceptually easy order, lex computations can be very long and tedious (even for a computer, if the size of the problem is big enough!). Other monomial orders are much more computationally efficient.

- Here's another example. Let's say we have monomials $x^{a_1}y^{b_1}z^{c_1}$ and $x^{a_2}y^{b_2}z^{c_2}$. We first compare $x^{a_1}y^{b_1}$ and $x^{a_2}y^{b_2}$ using grlex, and if there's a tie, only then do we check to see whether c_1 or c_2 is bigger. For example, under this order, we would have the following comparisons:

  $x^2yz^3 > xyz^{10}$ [since $x^2y >_{\text{grlex}} xy$]

  $x^2yz^3 > xy^2z^3$ [since $x^3y >_{\text{grlex}} xy^2$]

  $x^2yz^3 > x^2yz^2$ [since $x^2y = x^2y$ and $z^3 > z^2$]

  This is also of 2-elimination type: you should be able to see that, any monomial that involves x or y must be bigger than any monomial which involves z alone!

There are other examples of orders of 2-elimination type (some are described in the exercises). Here's the statement of the generalized elimination theorem in the special case where we have $n = 3$ variables and we're looking at orders of 2-elimination type.

Statement. Suppose I is an ideal in $k[x, y, z]$ and G is a Gröbner basis of I with respect to a monomial order of 2-elimination type, then $G_2 = G \cap k[z]$ is a Gröbner basis for the 2nd elimination ideal $I_2 = I \cap k[z]$.

Before proving this statement, let's work through an example to see what this is saying.

Example. Let's say we're looking at the ideal

$$I = \langle x^2 + y^2 + 1, y^2 + z^2 + 1, x \rangle$$

and we want to use the monomial order described in the second bullet point above. We set this up in Sage as follows.

```
R.<x,y,z> = PolynomialRing(QQ, order='deglex(2),lex(1)')
I = Ideal(x^2+y^2+1,y^2+z^2+1,x)
```

Now observe that some polynomials in the ideal I involve only z (no x or y). For example, we have

$$z^3 = z \cdot (y^2 + z^2 + 1) - z \cdot (x^2 + y^2 + 1) + xz \cdot x$$

so $z^3 \in I$. In other words, since z^3 is an element of both I and $k[z]$, it is an example of an element of the 2nd elimination ideal $I_2 = I \cap k[z]$. The statement of the (generalized) elimination theorem is that there's a systematic way of finding a (Gröbner) basis of I_2 as an ideal in $k[z]$ using a Gröbner basis for I. In other words, we can give an explicit description of all elements of I_2.

Let's compute a Gröbner basis for I using the command `I.groebner_basis()`. This outputs:

```
[y^2 + 1, x, z^2]
```

In other words, $G = \{y^2 + 1, x, z^2\}$ is a Gröbner basis for I. The statement above says that $G_2 = G \cap k[z]$ is a Gröbner basis for I_2. We inspect this Gröbner basis for elements which lie in $k[z]$, and only one of them does:

$$G_2 = \{z^2\}.$$

So the statement says that $G_2 = \{z^2\}$ is a Gröbner basis for I_2.

In particular, this means that I_2 is generated by z^2 as an ideal in $k[z]$, so

$$I_2 = \left\{ \sum a_n z^n \in k[z] \;\middle|\; a_0 = a_1 = 0 \right\}.$$

We've found an explicit description of elements of I_2, just by finding a Gröbner basis for I.

Proof of the statement. We know that $G_2 \subseteq I_2$, so $\langle LT(G_2) \rangle \subseteq \langle LT(I_2) \rangle$ (as ideals in $k[z]$). We want to show the opposite inclusion.

Suppose $f \in LT(I_2)$. Since $I_2 = I \cap k[z]$, this means that $f \in I$, and since G is a Gröbner basis for I, there exists $g \in G$ such that $LT(g)$ divides $LT(f)$. Since $f \in k[z]$, we know that $LT(f)$ only involves z, which means that $LT(g)$ also only involves z (otherwise it could not divide a monomial that involves only z!). But any monomial in g involving x or y would be bigger than a monomial involving z alone, so the fact that $LT(g)$ only involves z means that actually *all of the terms* of g only involve z. In other words, we have $g \in k[z]$. Thus $g \in G \cap k[z] = G_2$.

Thus we have just shown that, for any $f \in I_2$, the leading term $LT(f)$ is divisible by $LT(g)$ for some $g \in G_2$. This shows that $\langle LT(I_2) \rangle \subseteq \langle LT(G_2) \rangle$, which completes the proof. $\square$

I now encourage you to go back and try to write out the fully general solution to exercise 3.1.5, using the above as a model.