> Can you give an explanation of comprehension check, problem one?

Observe that

$$g = y(x - yz^4) - (xy - z^2) = -y^2z^4 + z^2$$

is an element of I. But its leading term with respect to lex order is $-y^2z^4$, and this is *not* divisible by $\mathrm{LT}(xy^2 - xz + y) = xy^2$ or by $\mathrm{LT}(xy - z^2) = xy$ or by $\mathrm{LT}(x - yz^4) = x$. So $\mathrm{LT}(g) \notin \langle \mathrm{LT}(g_1), \mathrm{LT}(g_2), \mathrm{LT}(g_3) \rangle$.

> [In the definition of Gröbner bases,] why do we use $\mathrm{LT}(I)$ but not $\mathrm{LM}(I)$?

It would be equivalent to use $\mathrm{LM}(I)$. In other words, one could equally well have defined a subset $\{g_1, \dots, g_t\}$ of an ideal I to be a Gröbner basis if $\langle \mathrm{LM}(I) \rangle = \langle \mathrm{LM}(g_1), \dots, \mathrm{LM}(g_t) \rangle$. This is essentially because the leading term and the leading monomial of a polynomial differ only by a nonzero constant, so it makes no difference which is used to generate an ideal. Exercise 2.5.4 asks you to make this precise.

To be honest, I would also prefer to define Gröbner bases using LM. It makes it clearer that the ideal $\langle \mathrm{LM}(S) \rangle$ generated by the set

$$\mathrm{LM}(S) = \{\mathrm{LM}(f) \mid f \in S \setminus \{0\}\}$$

is a monomial ideal for any subset $S \subseteq k[x_1, \dots, x_n]$. I don't know why the textbook authors chose to use LT instead.

> Is there a "descending" chain for ideals or does this only apply to varieties like in exercise 2.5.13?

There's a long story here! Let me sketch some of the salient points of this story.

First, if R is any ring, you can talk about the ascending chain condition and the descending chain condition on ideals in R. The ring R satisfies the *ascending chain condition* if every ascending chain of ideals

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$$

eventually stabilizers. A ring which satisfies the ascending chain condition is called *noetherian*. Similarly, R satisfies the *descending chain condition* if every descending chain of ideals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

eventually stabilizes, and a ring which satisfies the descending chain condition is called *artinian*.

While it looks like these two conditions are dual, the situation is *far* from being symmetric. Many important examples of rings are noetherian; in fact, *most* rings that you'll ever run into in your mathematical life will be noetherian. We've seen that polynomial rings $k[x_1, \dots, x_n]$ are noetherian. It's also not very hard to prove that the ring of integers $\mathbb{Z}$ is noetherian.

But most rings are *not* artinian! Perhaps $\mathbb{Z}$ is one of the simplest rings, but it is not artinian. An example of a descending chain of ideals in $\mathbb{Z}$ that never stabilizes is:

$$\langle 2\rangle \supsetneq \langle 4\rangle \supsetneq \langle 8\rangle \cdots$$

Polynomial rings are also not artinian — not even in one variable! For example,

$$\langle x\rangle \supsetneq \langle x^2\rangle \supsetneq \langle x^3\rangle \cdots$$

is a descending chain of ideals in $k[x]$ which never stabilizes!

But there are some artinian rings out there. For example, any field is artinian. We won't run into any artinian rings besides fields in this class, but they exist and it's not terribly surprising to run into artinian rings.

Weirdly enough, artinian rings actually must be noetherian (even though the converse is *far* from being true, as we've just seen).