> For the proof of Lemma 3, I see how the implications (iii) $\implies$ (ii) $\implies$ (i) are trivial, but I do not quite understand how (ii) $\implies$ (iii) is trivial.

Observe that any polynomial $f \in k[x_1, \ldots, x_n]$ is of the form

$$f = c_1 x^{\alpha(1)} + \cdots + c_s x^{\alpha(s)}$$

where $\alpha(1), \ldots, \alpha(s) \in \mathbb{Z}^n_{\geqslant 0}$ and $c_1, \ldots, c_s \in k$ are all nonzero. If every term of f lies in I, by definition of "term" this means that $c_i x^{\alpha(i)} \in I$ for all i. Since $c_i \neq 0$ and k is a field, it has an inverse c_i^{-1}. Since ideals are stable under multiplication by arbitrary elements of your ring, this means that

$$x^{\alpha(i)} = c_i^{-1} \cdot c_i x^{\alpha(i)} \in I.$$

Thus f is a k-linear combination of monomials in I. This proves (ii) $\implies$ (iii).

> I don't quite understand 3(b) in CC.

Remember that the picture you draw of the ideal $I = \langle x^6, x^2y^3, xy^7 \rangle$ tells you precisely what monomials are in I, or, equivalently, what monomials are divisible by at least one of the three monomials x^6, x^2y^3, xy^7.

When you apply the division algorithm to divide a polynomial by these three monomials, none of the terms of the remainder are divisible by any of these three monomials. This is part of the statement of the division algorithm in theorem 2.3.3, but you if you think about how the division algorithm works, remember that terms of the dividend only get moved over to the remainder column when they're not divisible by the leading terms of the divisors; since our divisors here are monomials, they are their own leading terms, so this means that terms of the dividend only get moved over to the remainder column when they're not divisible by any of the three monomials!

Putting all of this together, this means that all of the monomials that appear in the remainder must be in the *complement* of the shaded region that you drew for part (a)!

> I'm unsure as to what Dickson's lemma is useful for. I get that it lets us say that monomial ideals have a finite basis, but after that I'm unclear.

That is in fact the main use for now, but in your next reading, you'll see that we can use the fact that any monomial ideal is finitely generated to prove that *any ideal* in the polynomial ring $k[x_1, \ldots, x_n]$ is finitely generated! This is called the "Hilbert basis theorem" and it has far-reaching consequences; it'll come up repeatedly as we proceed through the course.