Most of you had roughly the right idea for 2.4.1, but had trouble formalizing it. Here is how I would write this proof up formally. Observe that there are three structural components to my proof below:

- Giving a precise description of a set A.
- Showing that $\langle x^\alpha : \alpha \in A \rangle \subseteq I$.
- Showing that $I \subseteq \langle x^\alpha : \alpha \in A \rangle$.

While a correct proof may look somewhat different than the below, it should address all three of these points in some way.

2.4.1. Suppose I is an ideal with the property that every monomial of every element of I is also in I. We want to show that I is a monomial ideal. Let

$$A = \{\alpha \in \mathbb{Z}^n_{\geqslant 0} \mid x^\alpha \in I\}.$$

In other words, A is the set of monomials that are in I. Let us show that $I = \langle x^\alpha \mid \alpha \in A \rangle$. Since $x^\alpha \in I$ for all $\alpha \in A$, it is clear that $\langle x^\alpha \mid \alpha \in A \rangle \subseteq I$. For the reverse inclusion, suppose $f \in I$. We can write f as

$$f = c_1 x^{\alpha(1)} + \cdots + c_n x^{\alpha(s)}$$

where $c_1, \ldots, c_s \in k$ are all nonzero and $\alpha(1), \ldots, \alpha(s) \in \mathbb{Z}^n_{\geqslant 0}$. Since I has the property that every monomial of any element of I is again in I, we have $x^{\alpha(i)} \in I$ for all $i = 1, \ldots, s$. But then $\alpha(i) \in A$ by definition of A, so f is a linear combination of monomials of the form x^α where $\alpha \in A$. This shows that $I \subseteq \langle x^\alpha \mid \alpha \in A \rangle$. $\square$

Similarly, there was some trouble with formalization for 2.4.2. Here is how I would write this up.

2.4.2. Suppose I is a monomial ideal and $f \in I$. We want to show that every term of f lies in I. By definition of monomial ideals, there exists a set $A \subseteq \mathbb{Z}^n_{\geqslant 0}$ such that $I = \langle x^\alpha \mid \alpha \in A \rangle$. This means that there exist $\alpha(1), \ldots, \alpha(s) \in A$ such that

$$f = h_1 x^{\alpha(1)} + \cdots + h_s x^{\alpha(s)} = \sum_{i=1}^{s} h_i x^{\alpha(i)}$$

where $h_1, \ldots, h_s \in k[x_1, \ldots, x_n]$. For each i, we can write $h_i = 1, \ldots, s$ as a linear combination of monomials:

$$h_i = c_{i,1} x^{\beta(i,1)} + \cdots + c_{i,t_i} x^{\beta(i,t_i)} = \sum_{j=1}^{t_i} c_{i,j} x^{\beta(i,j)}$$

where $c_{i,j} \in k$ are all nonzero. Then

$$f = \sum_{i=1}^{s} h_i x^{\alpha(i)} = \sum_{i=1}^{s} \sum_{j=1}^{t_s} c_{i,j} x^{\beta(i,j)+\alpha(i)}.$$

All terms appearing in this sum are divisible by $x^\alpha(i)$ for some i (more specifically, $x^{\beta(i,j)+\alpha(i)}$ is divisible by $x^{\alpha(i)}$). Applying lemma 2, this means that all terms of f are in I, which is what we wanted to prove. $\square$