Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x, y) = |xy|.$$

Recall that f is differentiable at the origin.

1. True or False?

There exists an open neighborhood U of the origin such that $f|_U$ is differentiable.

2. True or False?

Let $U = \{(x, y) : x > 0\}$ and let $f : U \to \mathbb{R}^2$ be the function

$$f(x, y) = (xe^{-y}, xe^{y}).$$

Then f is étale.

2. True or False?

Let $U = \{(x, y) : x > 0\}$ and let $f : U \to \mathbb{R}^2$ be the function

$$f(x, y) = (xe^{-y}, xe^{y}).$$

Then f is étale.

Follow-up. Show that f is injective. Observe that $f(1, 0) = (1, 1)$. What is $(f^{-1})'(1, 1)$? What is $f(U)$?

Recall that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$$

has the property that the partial derivatives exist everywhere, but f is discontinuous at 0.

3. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that $\partial_1 f$ and $\partial_2 f$ both exist, and suppose further that there exists a constant $M > 0$ such that $|\partial_1 f(x)|, |\partial_2 f(x)| \leq M$ for all $x \in \mathbb{R}^2$. Prove that f is continuous.

Tedious, but important... Sorry!

4. Calculate the degree 2 Taylor polynomial of

$$f(x,y) = \ln(x^2 + y^2)$$

at the point $(1,0)$.

5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function with all of the following properties.

- $f(0) = 0$.
- 0 is a critical point of f.
- $\partial^{(1,1)} f(0) = 0$.
- $\partial^{(2,0)} f(0),\ \partial^{(0,2)} f(0) > 0$.

Prove that 0 is a local min of f.

Follow-up. What happens if the last condition instead stated that $\partial^{(2,0)} f(0)$ and $\partial^{(0,2)} f(0)$ are both negative? What about if one is positive and the other is negative?