A Smörgåsbord of Arithmetic Geometry

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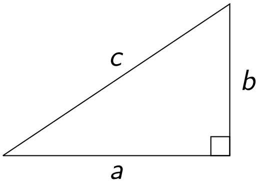
Smörgåsbord

Arithmetic geometry is an active area of mathematical research with a rich history. Today, I'd like to give you a taste of the field with a smörgåsbord of motivating examples.

Outline

The Pythagorean theorem



If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then

$$a^2 + b^2 = c^2.$$

Pythagorean triples

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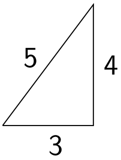

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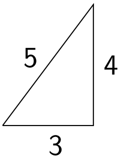

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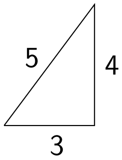

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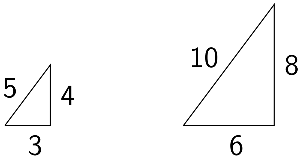
$$6^2 + 8^2 = 36 + 64 = 100 = 10^2.$$



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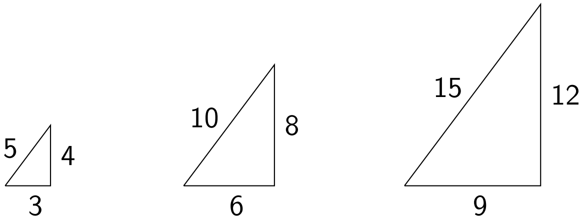

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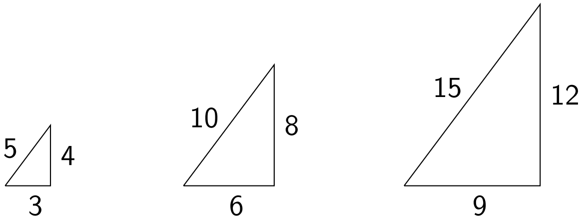

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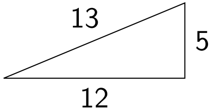

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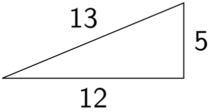

As before, its integer multiples (like $(24, 10, 26), (36, 15, 39), \ldots$) are also Pythagorean triples.

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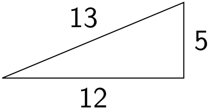

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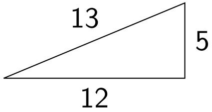

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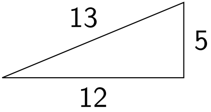

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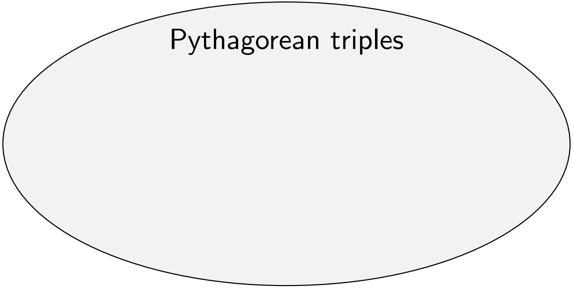
For example, $(3, 4, 5)$ is reduced, but $(6, 8, 10)$ is not since 2 is a common factor of $6, 8$ and 10.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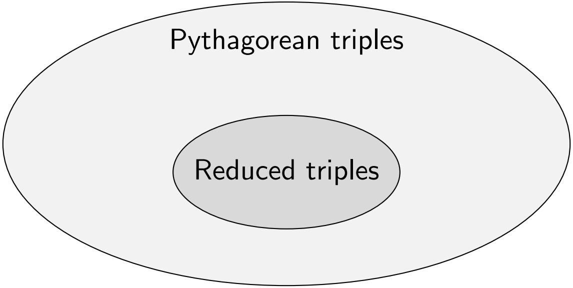
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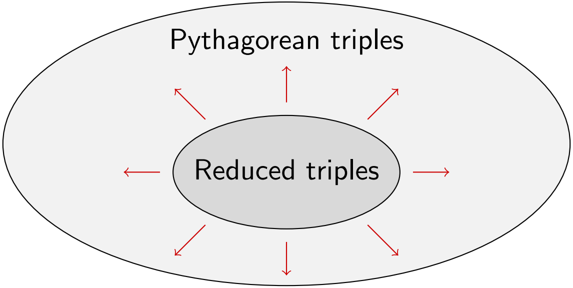

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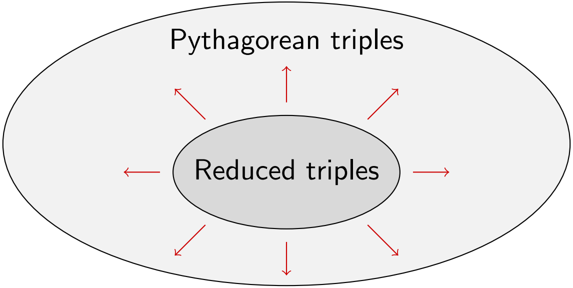

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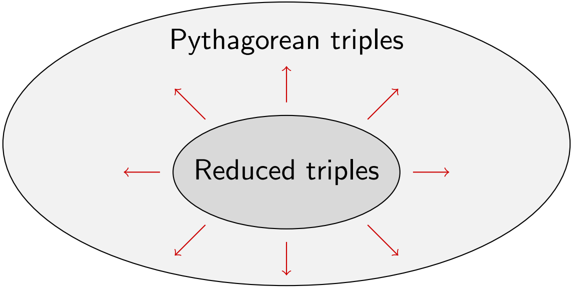

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Our goal will be to systematically list the reduced triples.

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If I add, subtract, multiply, or divide two rational numbers, the result will still be a rational number.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

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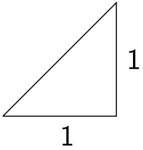

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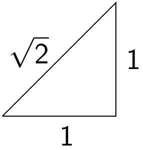

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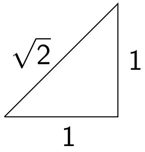

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It has been known for thousands of years that this number is *irrational*. A classical proof by contradiction can be found in Euclid's *Elements*.

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In other words, $x = a/c$ and $y = b/c$ are rational numbers satisfying

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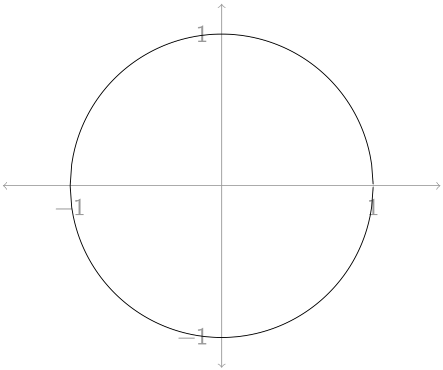
The equation $x^2 + y^2 = 1$ defines a circle of radius 1.



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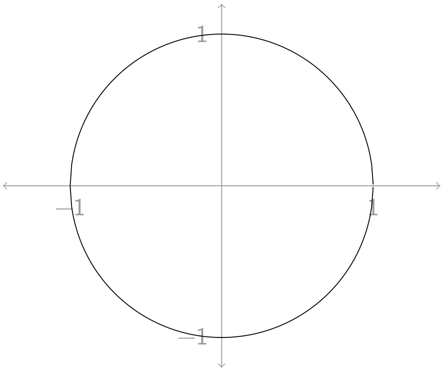

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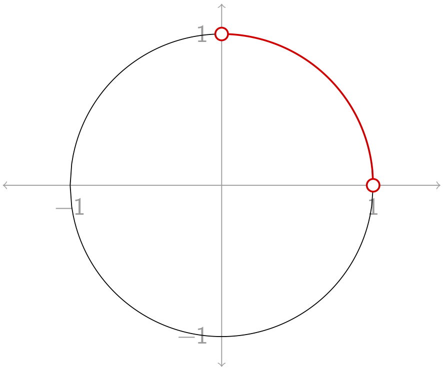
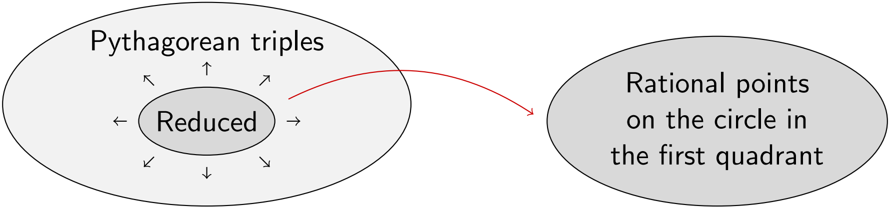
If a, b, c are all positive, so are $x = a/c$ and $y = b/c$...

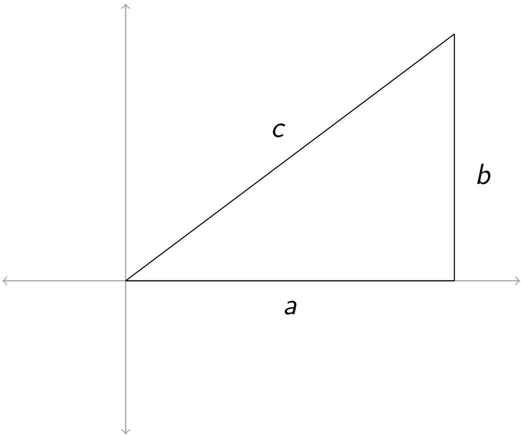
...so (x, y) is a *rational point* on the circle in the first quadrant.





We started with a reduced Pythagorean triple and found a rational point on the circle $x^2 + y^2 = 1$ inside the first quadrant.

Geometrically, here's what we did.



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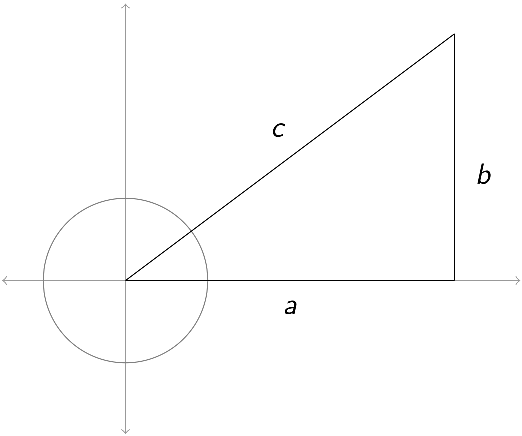

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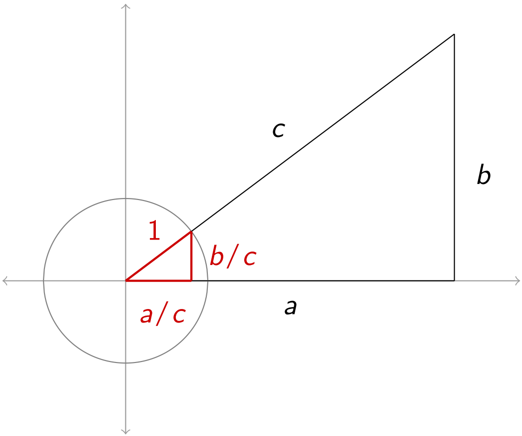

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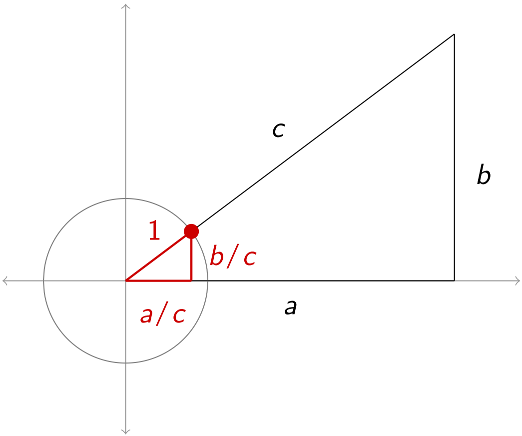

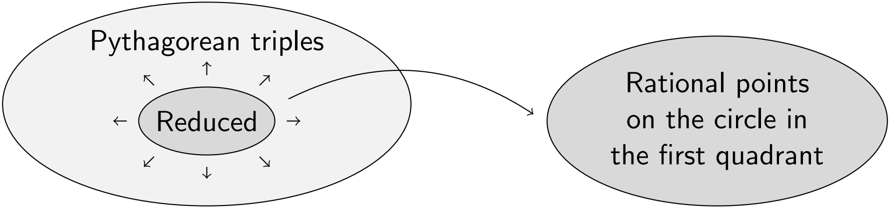
Pythagorean triples
Reduced
Rational points
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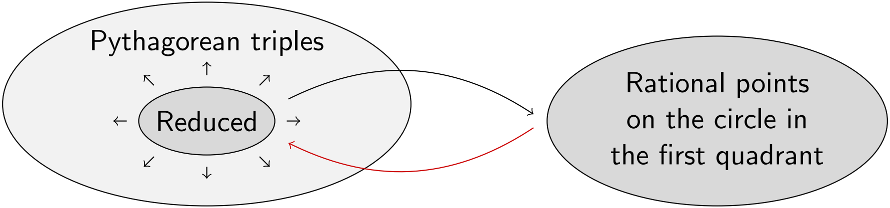

We can also go the other way: given any rational point on the circle $x^2 + y^2 = 1$ in the first quadrant, we can get a reduced Pythagorean triple.

Let's see how!

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For example,

$$\frac{1}{4} + \frac{5}{6} = \frac{3}{12} + \frac{10}{12} = \frac{13}{12},$$

and the least common denominator of $1/4$ and $5/6$ is 12.

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Since c is the least common denominator, (a, b, c) is a reduced Pythagorean triple.

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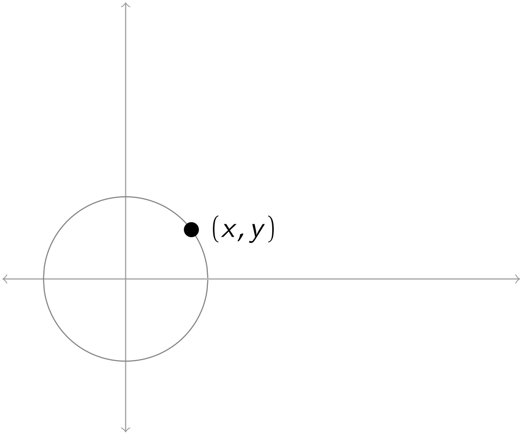

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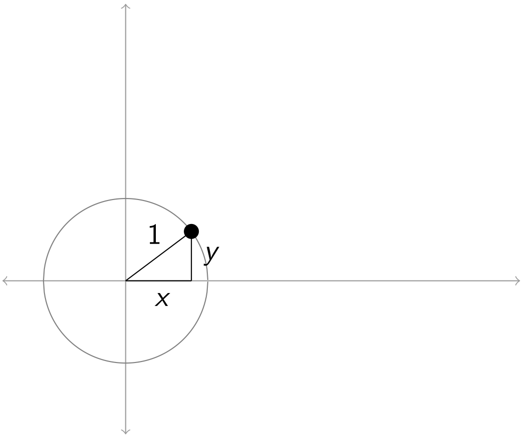

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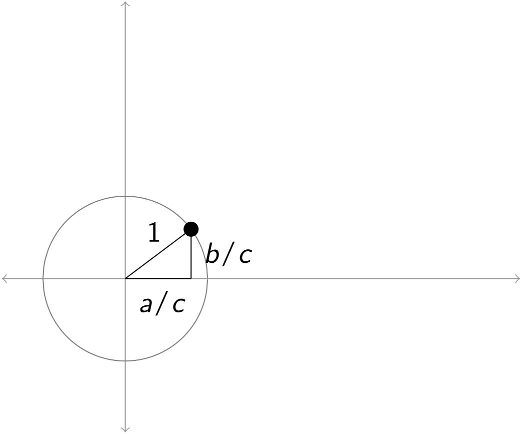

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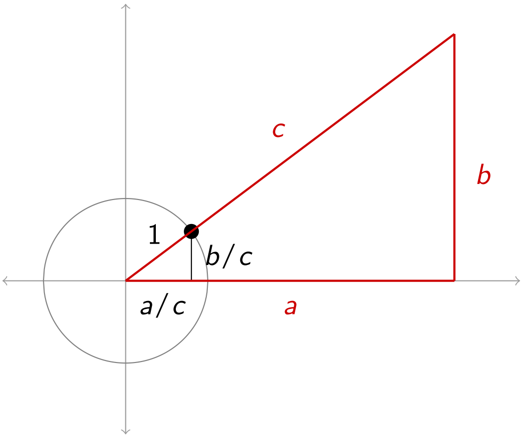
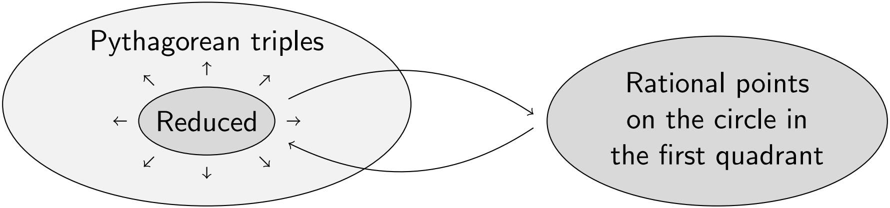
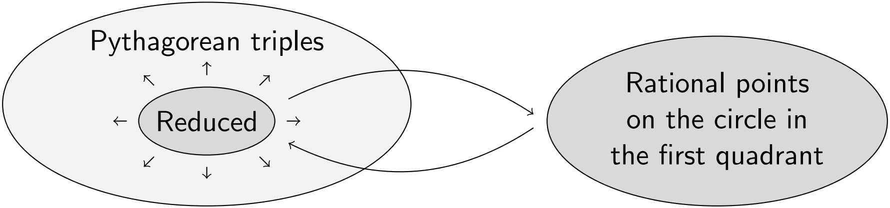




So, finding reduced Pythagorean triples is the same as finding rational points on the circle $x^2 + y^2 = 1$ inside the first quadrant.



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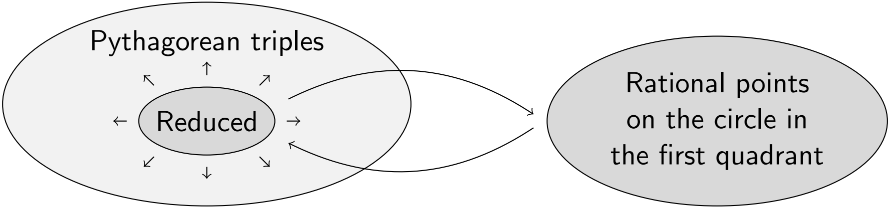

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Let's see!

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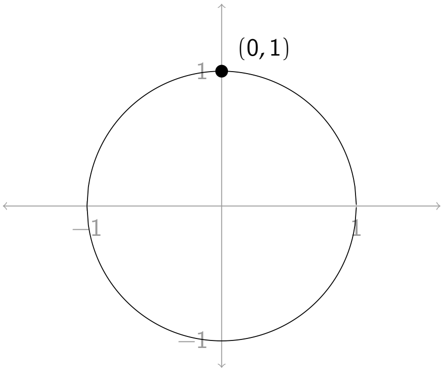
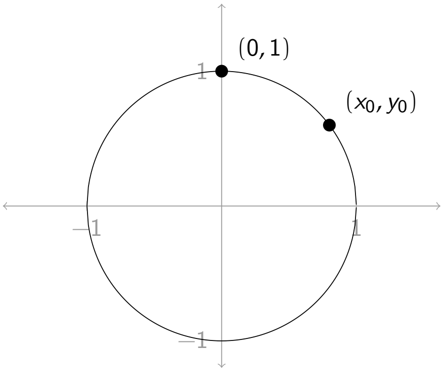


The point $(0, 1)$ is on the circle.

Let's say that (x_0, y_0) is another rational point on the circle.



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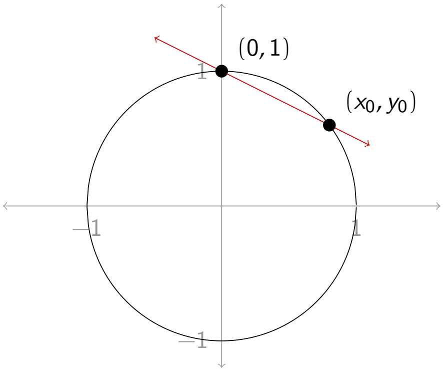
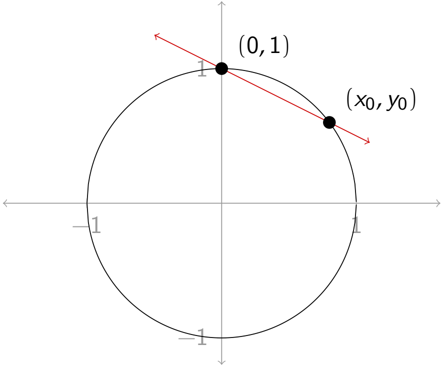
Draw a line through $(0,1)$ and (x_0, y_0).



This line has equation

$$y = \underbrace{\left(\frac{y_0 - 1}{x_0}\right)}_{r} x + 1.$$



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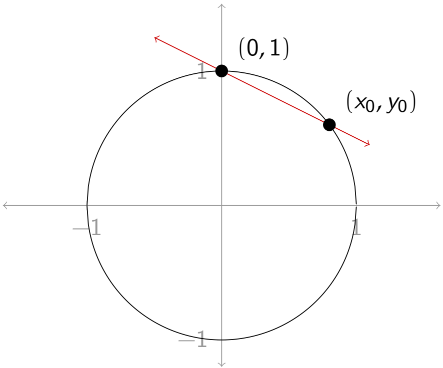
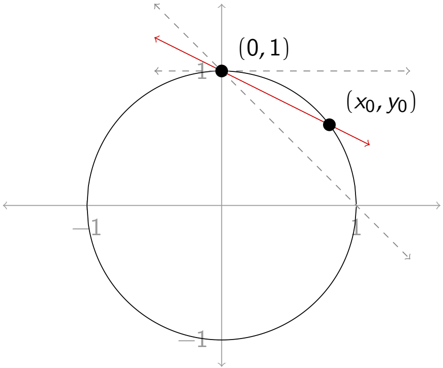
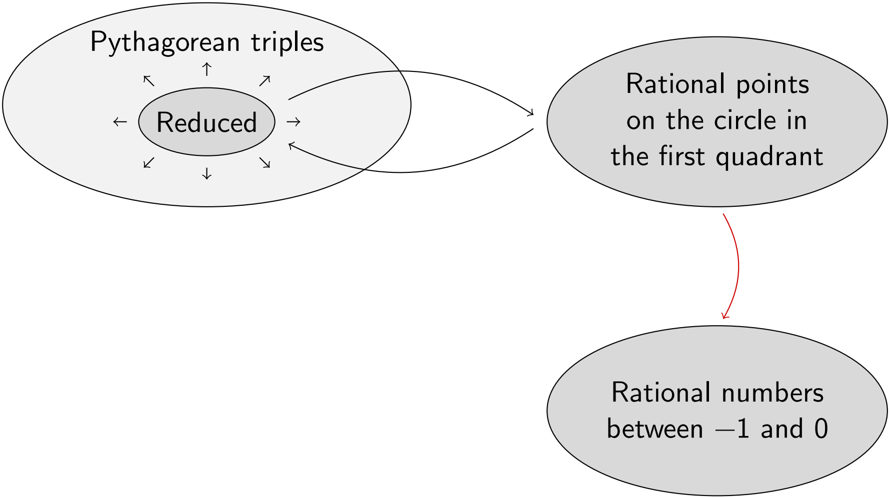
Notice that the slope r is a rational number.



If (x_0, y_0) is in the first quadrant, the slope r is between -1 and 0.

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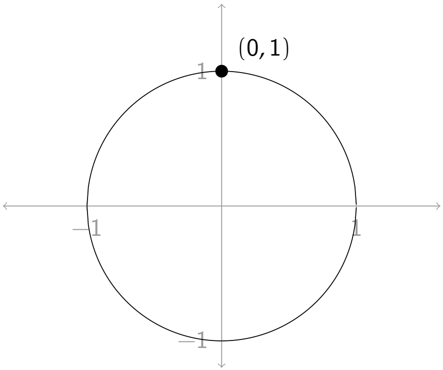

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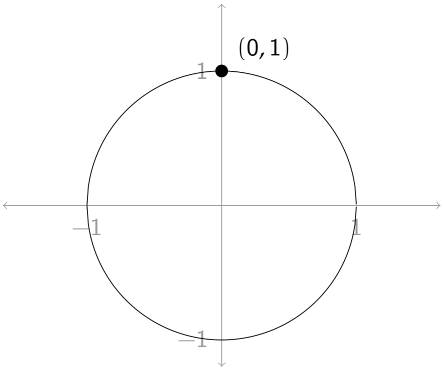
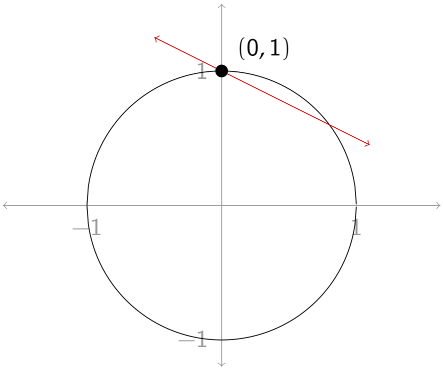
Suppose we start with an arbitrary rational slope r between -1 and 0...



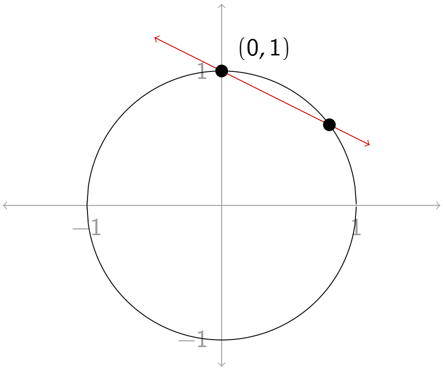
Can we go the other way?

Suppose we start with an arbitrary rational slope r between -1 and 0...

... and we draw the line of slope r passing through $(0, 1)$.

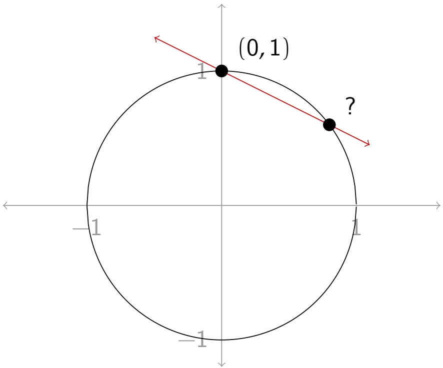


This line will intersect the circle in another point inside the first quadrant.



This line will intersect the circle in another point inside the first quadrant.

Will that point have rational coordinates?



Let's find the coordinates of the point of intersection.

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In other words, we want to solve the following system of equations.

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Substituting $y = rx + 1$ into $x^2 + y^2 = 1$, we find

$$x^2 + (rx + 1)^2 = 1.$$

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Since $y = rx + 1$, we know that y is rational when x is rational.

So the second point of intersection is a rational point!

We can find the coordinates of the second point of intersection explicitly.

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\begin{aligned}
x^2 + (rx + 1)^2 &= 1 \\
x^2 + (r^2x^2 + 2rx + 1) &= 1
\end{aligned}
$$

We can find the coordinates of the second point of intersection explicitly.

$$
\begin{aligned}
x^2 + (rx+1)^2 &= 1 \\
x^2 + (r^2x^2 + 2rx + 1) &= 1 \\
(1+r^2)x^2 + 2rx &= 0
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$$\begin{aligned} x^2 + (rx+1)^2 &= 1 \\ x^2 + (r^2x^2 + 2rx + 1) &= 1 \\ (1+r^2)x^2 + 2rx &= 0 \\ x((1+r^2)x + 2r) &= 0 \end{aligned}$$

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Note that $x = 0$ corresponds to the point $(0, 1)$, so we want $x = \dfrac{-2r}{1+r^2}$.

Plugging in $x = \dfrac{-2r}{1+r^2}$, we have

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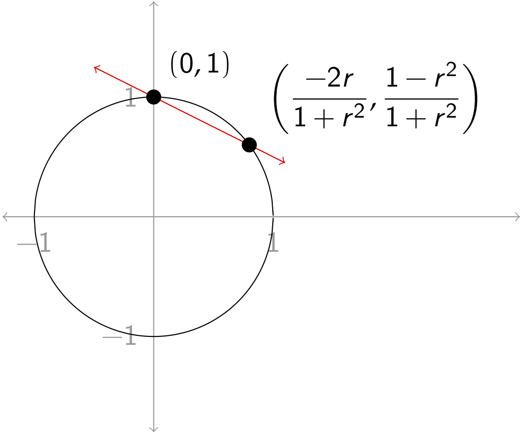
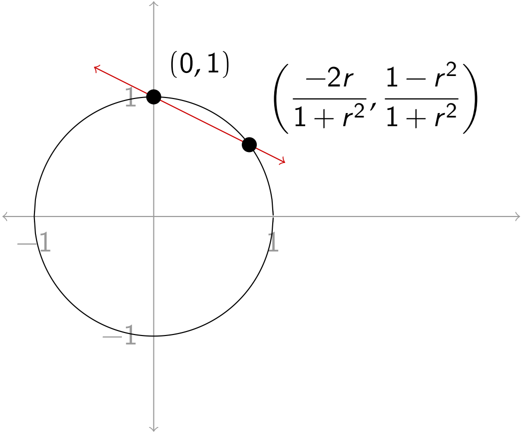
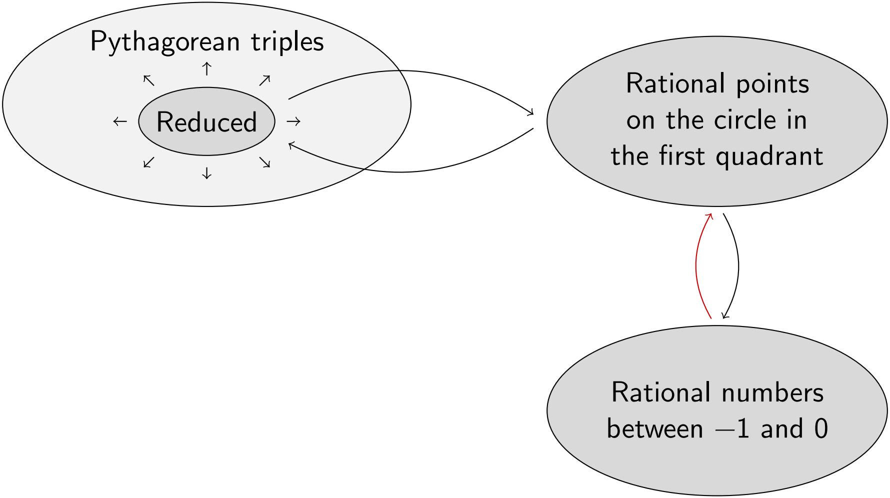


$(0, 1)$
$\left(\frac{-2r}{1+r^2}, \frac{1-r^2}{1+r^2}\right)$
1
-1
1
-1



Since r is rational, the second point of intersection is too!

What we've discovered!



Listing the rational numbers

We can systematically list off the rational numbers between -1 and 0 as follows:

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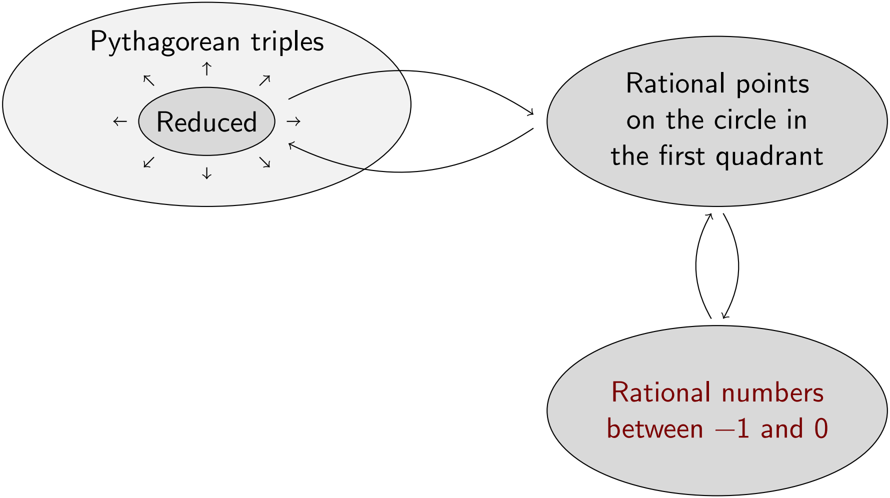
Generating the Pythagorean triples



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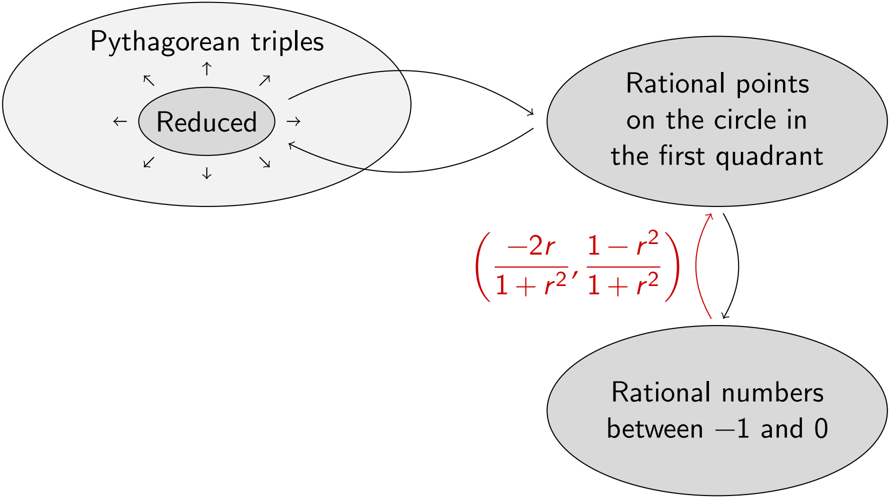

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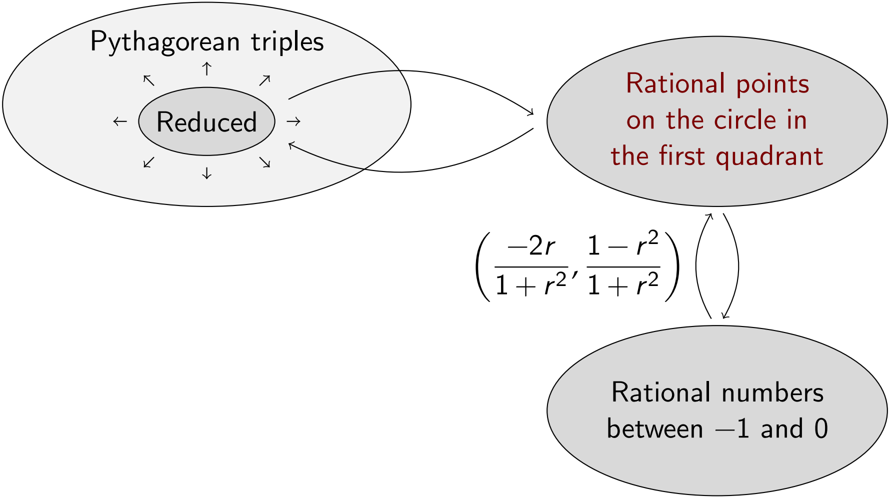

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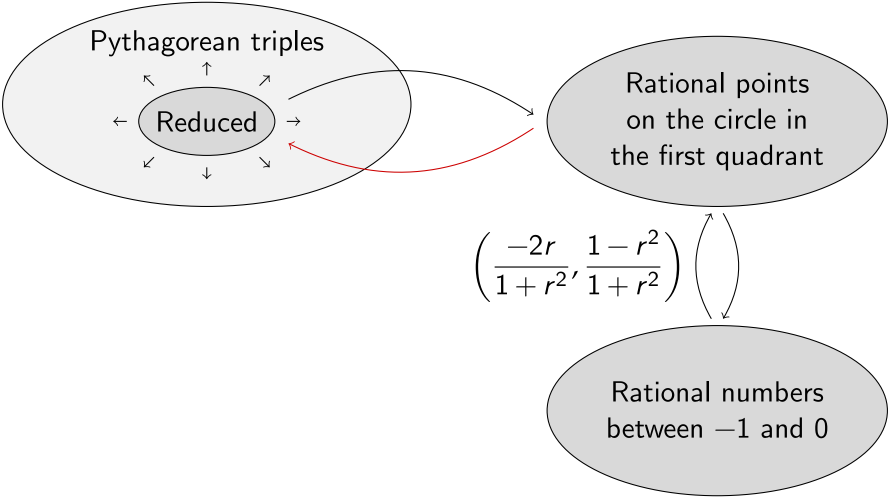

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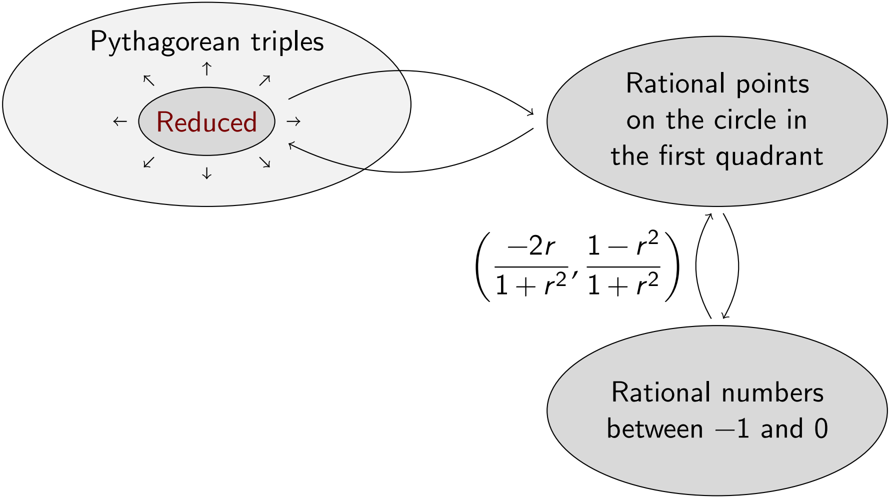

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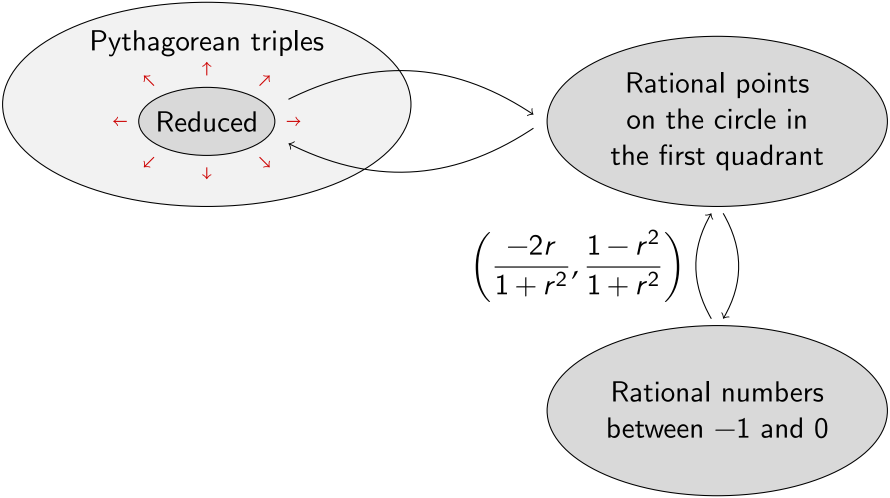

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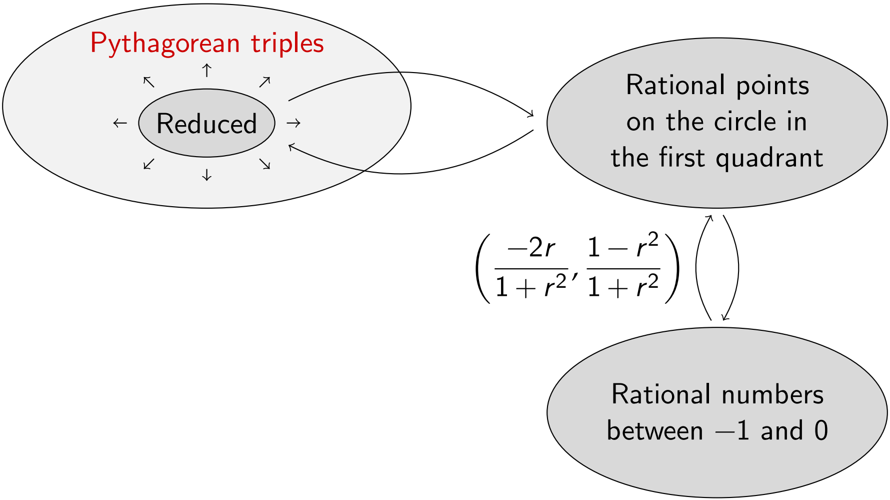
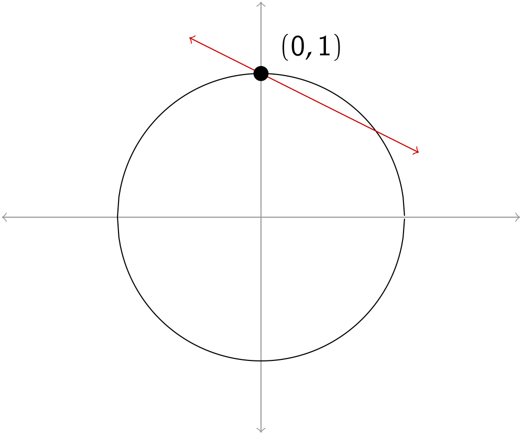
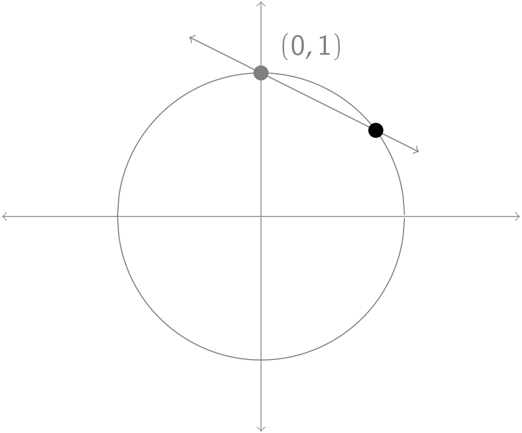


$$r = -\frac{1}{2}$$



$$r = -\frac{1}{2} \quad \rightsquigarrow \quad \left(\frac{1}{5/4}, \frac{3/4}{5/4}\right)$$



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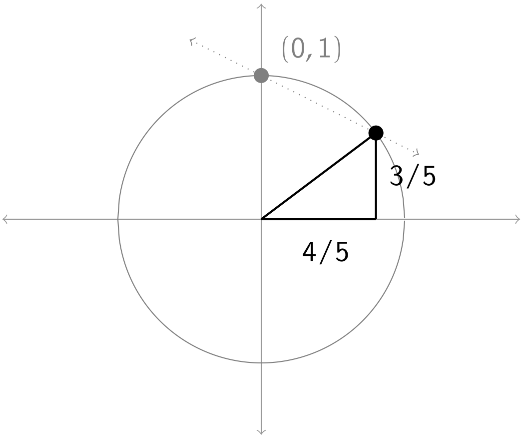
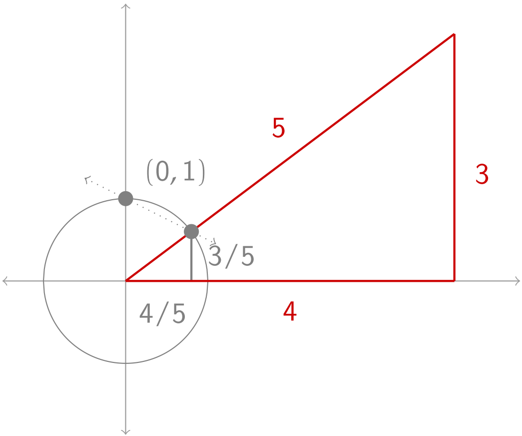


$$r = -\frac{1}{2} \quad \rightsquigarrow \quad \left(\frac{1}{5/4}, \frac{3/4}{5/4}\right) = \left(\frac{4}{5}, \frac{3}{5}\right) \quad \rightsquigarrow \quad (4, 3, 5)$$



Rational Slope		Rational Point		Reduced Triple
$-1/2$	$\rightsquigarrow$	$(4/5, 3/5)$	$\rightsquigarrow$	$(4, 3, 5)$

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$-1/3$				

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$-1/3$	$\rightsquigarrow$	$(3/5, 4/5)$		

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$-1/3$	$\rightsquigarrow$	$(3/5, 4/5)$	$\rightsquigarrow$	$(3, 4, 5)$
$-2/3$				

Rational Slope		Rational Point		Reduced Triple
$-1/2$	$\rightsquigarrow$	$(4/5, 3/5)$	$\rightsquigarrow$	$(4, 3, 5)$
$-1/3$	$\rightsquigarrow$	$(3/5, 4/5)$	$\rightsquigarrow$	$(3, 4, 5)$
$-2/3$	$\rightsquigarrow$	$(12/13, 5/13)$	$\rightsquigarrow$	$(12, 5, 13)$

Rational Slope		Rational Point		Reduced Triple
$-1/2$	$\rightsquigarrow$	$(4/5, 3/5)$	$\rightsquigarrow$	$(4, 3, 5)$
$-1/3$	$\rightsquigarrow$	$(3/5, 4/5)$	$\rightsquigarrow$	$(3, 4, 5)$
$-2/3$	$\rightsquigarrow$	$(12/13, 5/13)$	$\rightsquigarrow$	$(12, 5, 13)$
$-1/4$				

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$-2/3$	$\rightsquigarrow$	$(12/13, 5/13)$	$\rightsquigarrow$	$(12, 5, 13)$
$-1/4$	$\rightsquigarrow$	$(8/17, 15/17)$	$\rightsquigarrow$	$(8, 15, 17)$

Rational Slope		Rational Point		Reduced Triple
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$-2/4$				

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$-2/3$	$\rightsquigarrow$	$(12/13, 5/13)$	$\rightsquigarrow$	$(12, 5, 13)$
$-1/4$	$\rightsquigarrow$	$(8/17, 15/17)$	$\rightsquigarrow$	$(8, 15, 17)$
$\cancel{-2/4}$				
$-3/4$	$\rightsquigarrow$	$(24/25, 7/25)$	$\rightsquigarrow$	$(24, 7, 25)$

Rational Slope		Rational Point		Reduced Triple
$-1/2$	$\leadsto$	$(4/5, 3/5)$	$\leadsto$	$(4, 3, 5)$
$-1/3$	$\leadsto$	$(3/5, 4/5)$	$\leadsto$	$(3, 4, 5)$
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$-1/4$	$\leadsto$	$(8/17, 15/17)$	$\leadsto$	$(8, 15, 17)$
$\cancel{-2/4}$				
$-3/4$	$\leadsto$	$(24/25, 7/25)$	$\leadsto$	$(24, 7, 25)$
$\vdots$		$\vdots$		$\vdots$

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For example, starting with $r = -13711/31161$ (which is a rational number between -1 and 0),

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For example, starting with $r = -13711/31161$ (which is a rational number between -1 and 0), we get the Pythagorean triple

$$(472\,248\,471, 391\,508\,200, 579\,499\,721)$$

which you might not have known about.

Food for thought

When I wanted to find a really big Pythagorean triple, I chose the really crazy-looking fraction $r = -13711/31161$ instead of something like $r = -5/6$. Why might crazy-looking fractions give us big Pythagorean triples?

Outline

1. Pythagorean triples
2. The Hardy-Ramanujan number
3. Fermat's last theorem
4. What is arithmetic geometry?

Hardy and Ramanujan

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Ramanujan immediately responded, “No, Hardy! It is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways.”

And indeed, $1729 = 1^3 + 12^3 = 9^3 + 10^3$.

1729 as a sum of cubes

Ramanujan was talking about writing 1729 as the sum of cubes of two positive integers.

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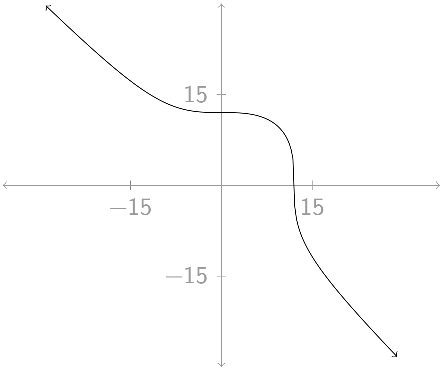
Let's think about the related problem of writing 1729 as the sum of cubes of two *rational* numbers.

In other words, we want to think about rational points on the *elliptic curve*

$$x^3 + y^3 = 1729.$$



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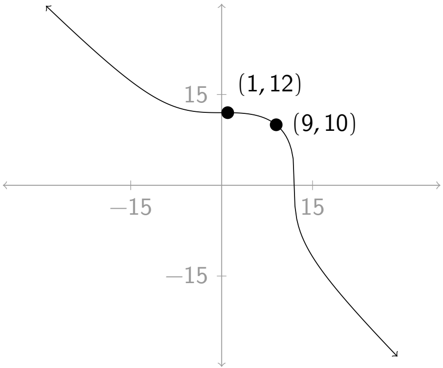
Ramanujan gave us two rational points on this elliptic curve.



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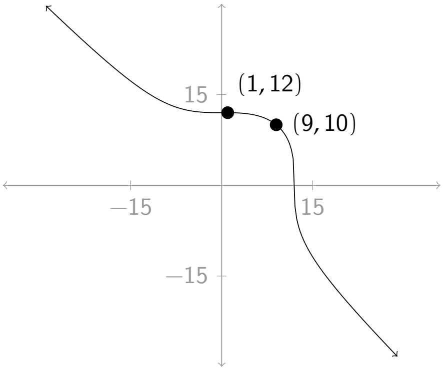
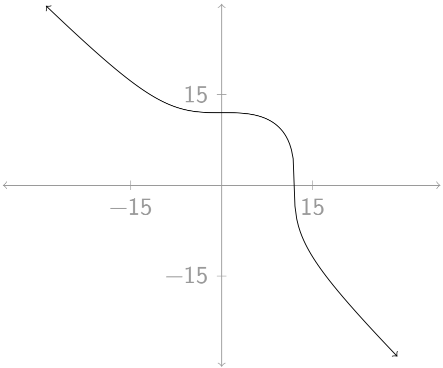
$$x^3 + y^3 = 1729.$$

Ramanujan gave us two rational points on this elliptic curve.

Is there a systematic way of producing all of the rational points?

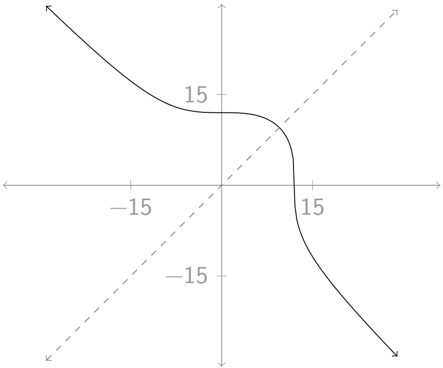


We have one really easy way of finding new rational points.



We have one really easy way of finding new rational points.

Our curve is symmetric about $y = x$.



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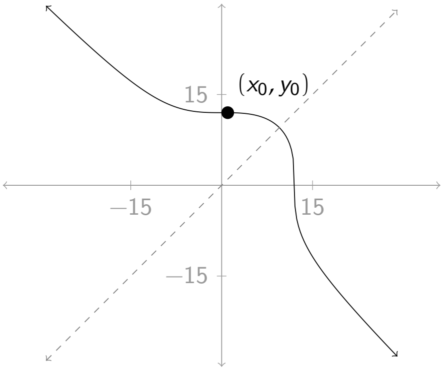
Our curve is symmetric about $y = x$.

So, if $P = (x_0, y_0)$ is a rational point,



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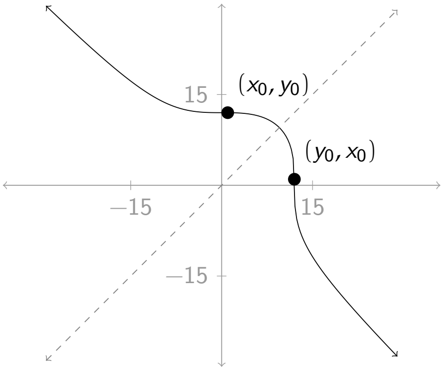
Our curve is symmetric about $y = x$.

So, if $P = (x_0, y_0)$ is a rational point, so is its reflection across this line.



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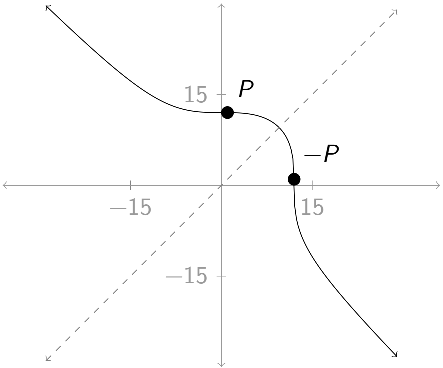
So, if $P = (x_0, y_0)$ is a rational point, so is its reflection across this line.

Let's call this reflected point $-P$.



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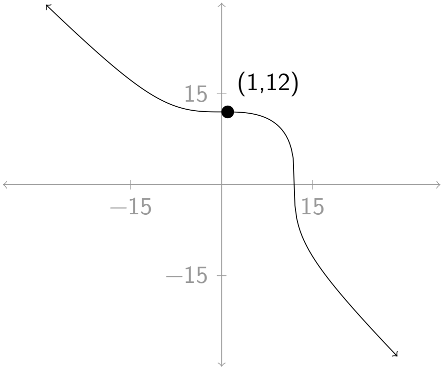
Previously our strategy for finding rational points was to first find one rational point and then to draw lines of rational slope through that point.

Let's try it!

Let's take the point $(1, 12)$...



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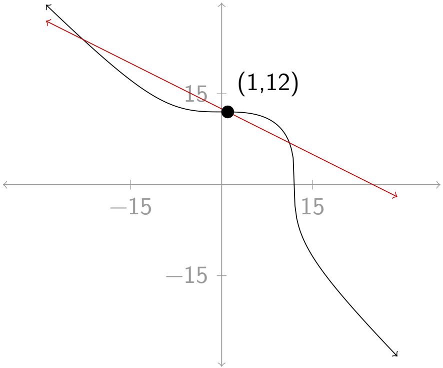

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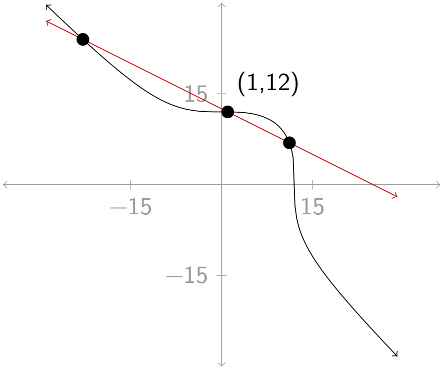
It intersects the curve in *two* other points,



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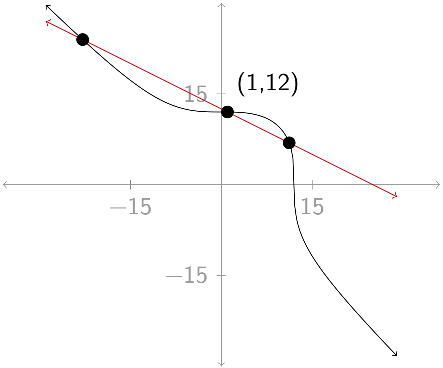

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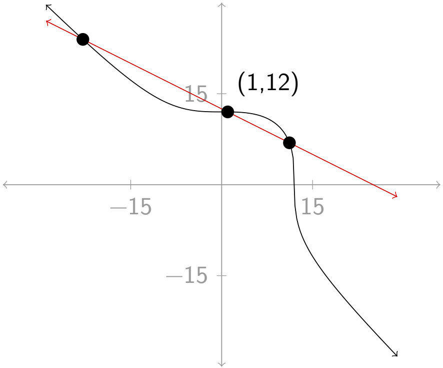

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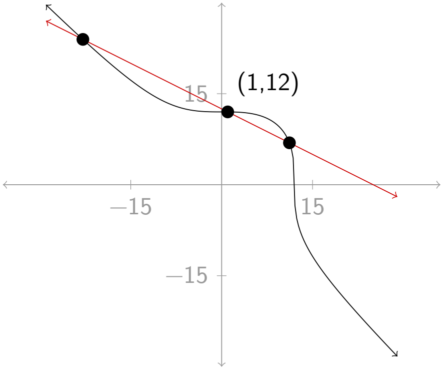

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The fact about polynomials with rational coefficients that we used earlier doesn't apply anymore.

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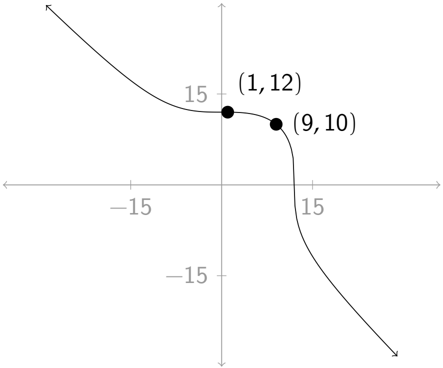
Let's insist that the line through $(1, 12)$ also pass through *another* rational point on the curve.

Thankfully, Ramanujan gave us another point: namely, $(9, 10)$.

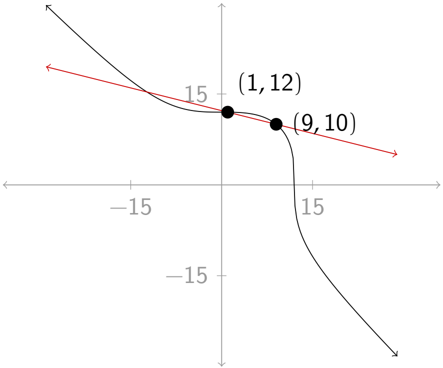
We have the two rational points $(1, 12)$ and $(9, 10)$.



We have the two rational points $(1, 12)$ and $(9, 10)$.

Draw the line that passes through them:

$$y = \frac{-x}{4} + \frac{49}{4}.$$



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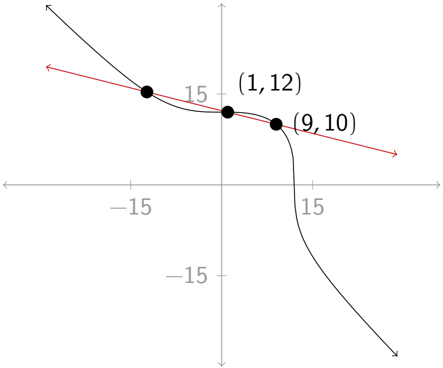

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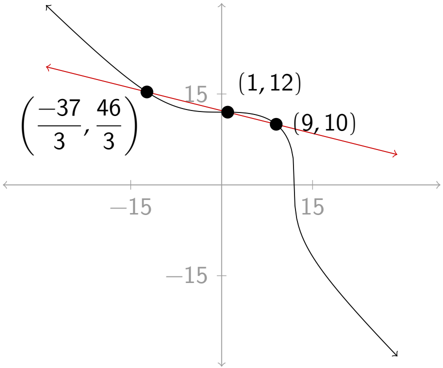

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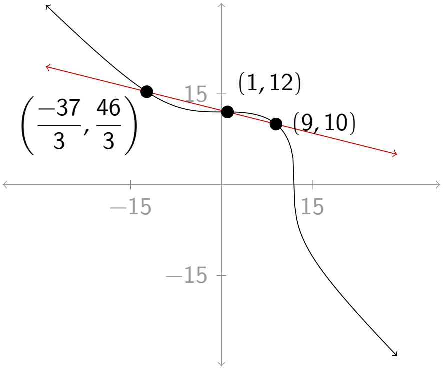
This is a rational point!



So we've discovered another way of writing 1729 as a sum of two rational cubes:

$$\left(\frac{-37}{3}\right)^3 + \left(\frac{46}{3}\right)^3 = 1729.$$

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Given two different rational points P and Q on the curve $x^3 + y^3 = 1729$, the line that goes through them will (usually) intersect the curve in another point, and this point must be rational.

Our argument that the third point of intersection must be rational was kind of abstract, but it is possible to write down its coordinates explicitly in terms of the coordinates of P and Q. Try it at home!

Snag

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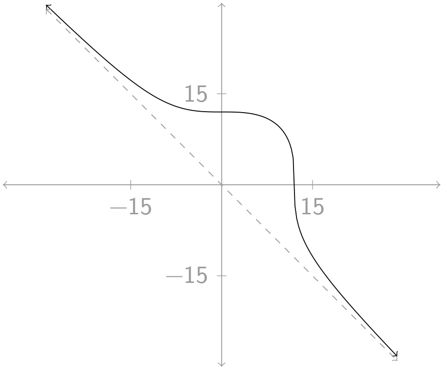
Notice that that $y = -x$ is a slant asymptote for the curve.



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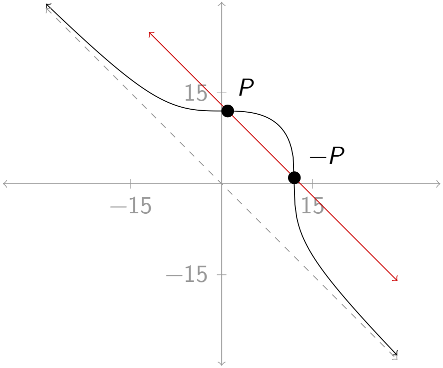

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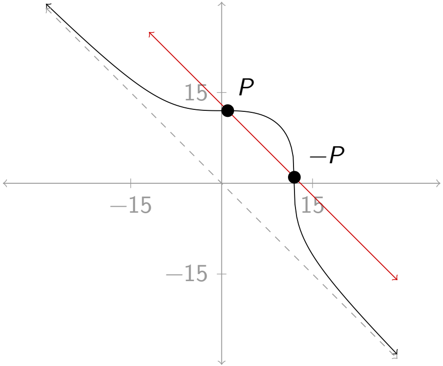
So there is no third point of intersection!



Fiat O!

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Now every line passing through two distinct rational points P and Q on the curve intersects the curve in a third rational point.

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This idea is the starting point for *projective geometry*.

“Adding” points

Addition of numbers is a nice way of taking two numbers and producing a third number.

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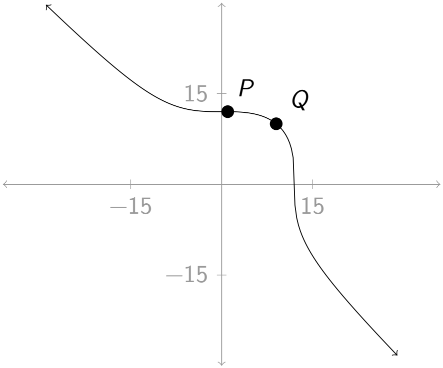
For the experts...

We are going to turn E into an abelian group with identity element O.

Addition on E

Start with any two distinct points P and Q.



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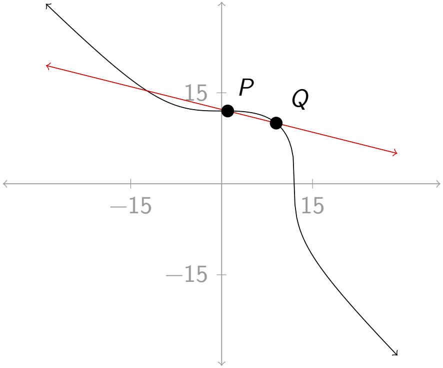

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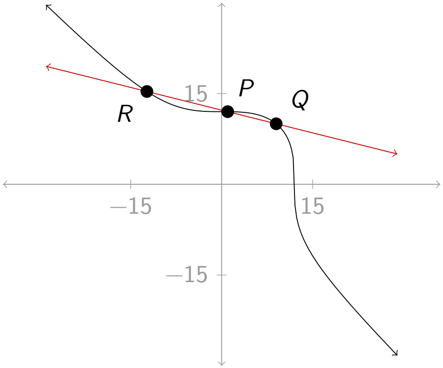

Addition on E

Start with any two distinct points P and Q.

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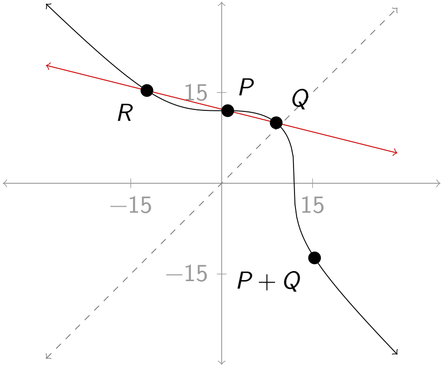
Then define $P + Q$ to be the reflection of R across the line $y = x$.



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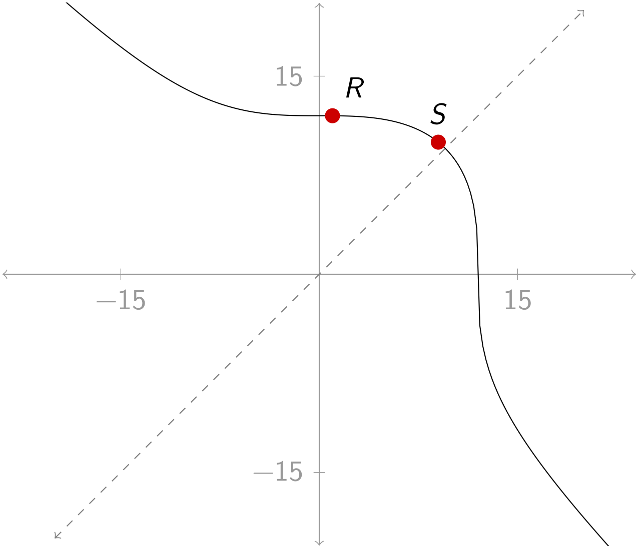
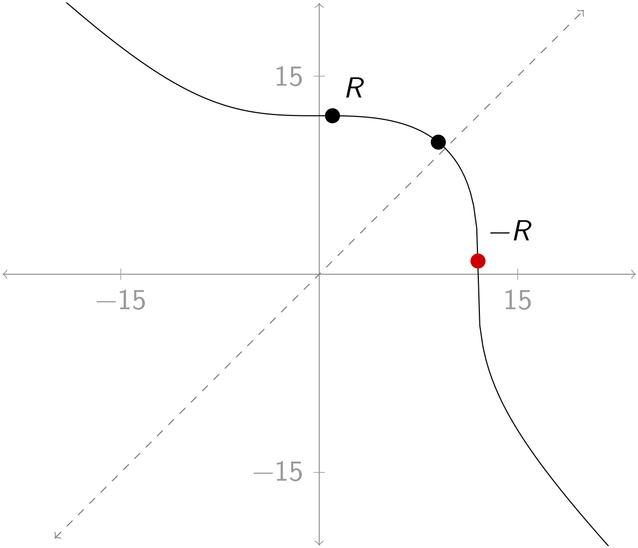
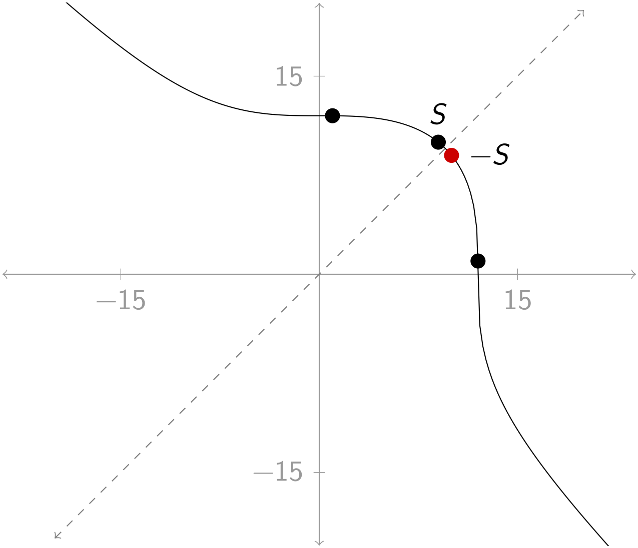
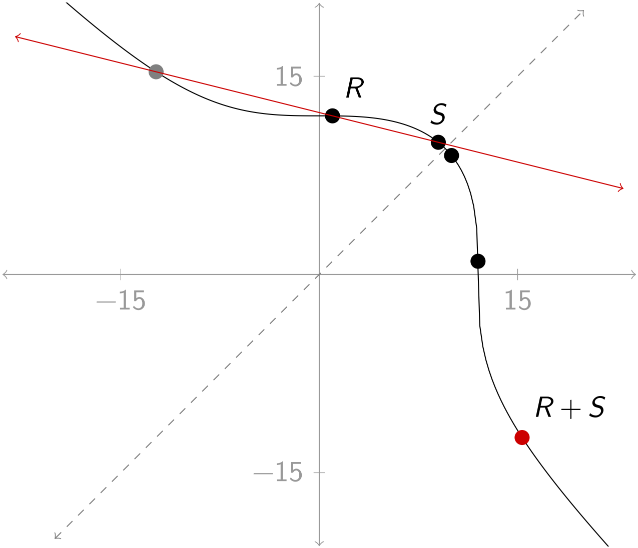
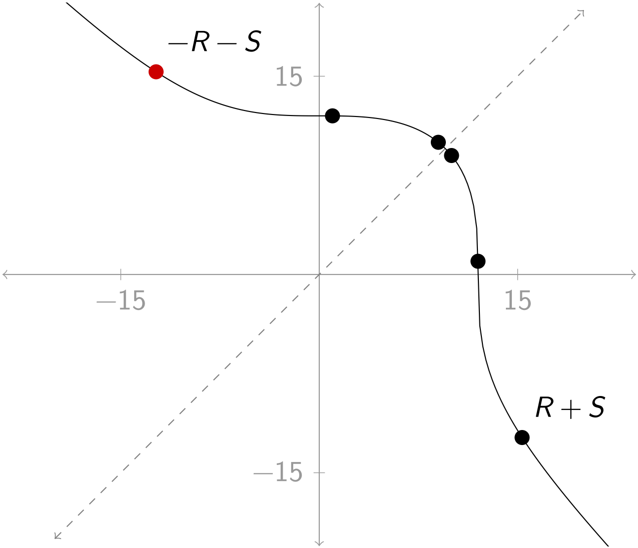
Do we generate all of the rational points on the curve this way?


15
R
S
−15
15
−15

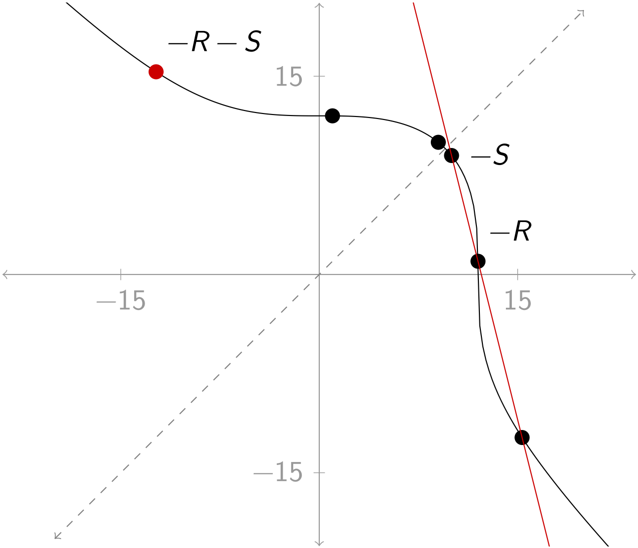

15
R
$-R$
-15
15
-15


15
−15
15
−15
S
−S

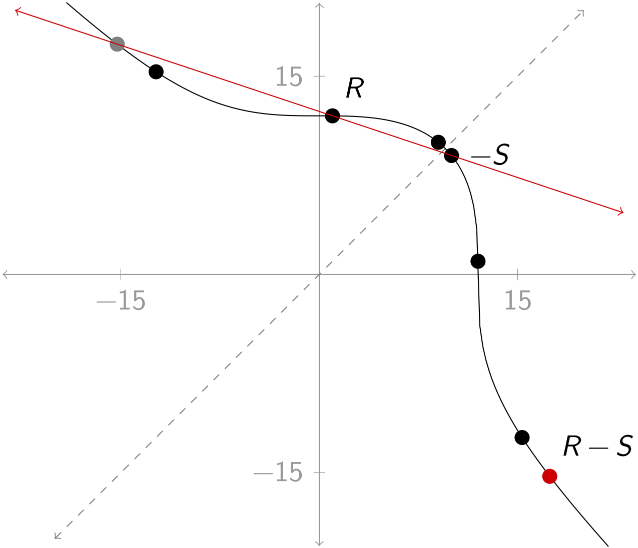

15
R
S
-15
15
$R+S$
-15



$-R-S$
15
-15
15
$R+S$
-15



$-R-S$
15
$-S$
$-R$
-15
15
-15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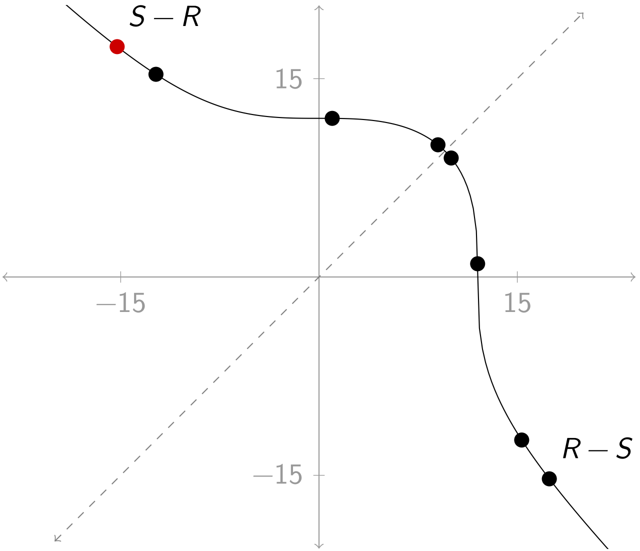
S − R
15
−15
15
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For the experts...

The Mordell-Weil theorem says that E is a *finitely generated* abelian group.

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- By using "3-descent," we learn that E has rank at most 2.

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Fortunately, we have computers!

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Ramanujan's points *do* generate all of the others!

Food for thought

- We defined $P+Q$ when P and Q are different. What should $P+P$ be? (Hint: $P+P=(P+Q)+(P-Q)$.)

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- Once you've worked out the answer to the previous question, explain why there is no $P \in E$ such that $P+P=O$.

Outline

1. Pythagorean triples
2. The Hardy-Ramanujan number
3. Fermat's last theorem
4. What is arithmetic geometry?

Fermat's last theorem

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He did, however, prove this for the exponent $n = 4$.

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This gives us a counterexample (a^m, b^m, c^m) to Fermat's last theorem with prime exponent p.

What about $p = 2$?

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I'll let you think about how to deal with this case. (Hint: Remember that Fermat proved the case $n = 4$.)

Century of stagnation

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Then, in 1770, Leonhard Euler published a proof of Fermat's last theorem with exponent $p = 3$.

Legendre and Dirichlet

 

Half a century later, in 1825, Adrien-Marie Legendre and Johann Peter Gustav Lejeune Dirichlet independently published proofs for $p = 5$.

Lamé

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But his proof was incorrect.

Lamé was not the only one: over the centuries, thousands of incorrect proofs of Fermat's last theorem have been proposed.

Kummer and the regular primes

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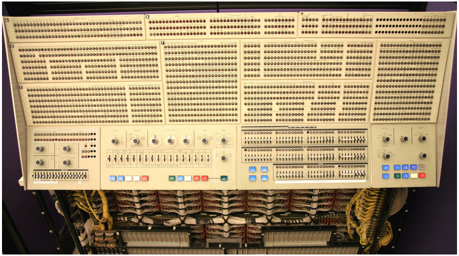
Most primes are regular: the smallest few irregular primes are

$$37, 59, 67, 101, 103, 131, \ldots .$$

Computational studies



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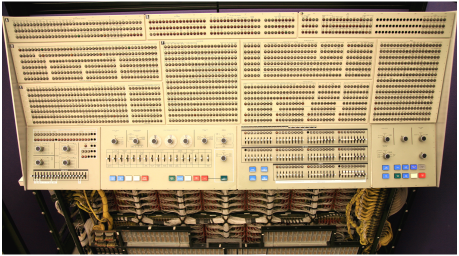

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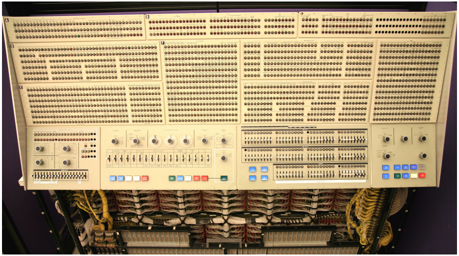

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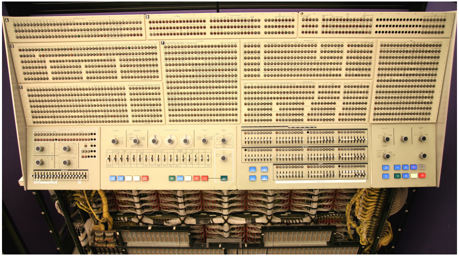

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- By 1978, all primes up to 124 991...
- And by 1993, all primes up to 4 000 000.

Enter arithmetic geometry.

Serre and Ribet

In 1985-86, Jean-Pierre Serre and Ken Ribet proved that, if (a, b, c) were a counterexample to Fermat's last theorem for some prime exponent $p \geq 5$, the elliptic curve

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would have some *very* strange properties...

Properties so strange, in fact, that it had been conjectured a few decades earlier that no elliptic curve could have those properties!

Wiles

In 1995, Andrew Wiles proved that the elliptic curve

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was not as strange as it would have to be for (a, b, c) to be a counterexample to Fermat's last theorem with prime exponent $p \geq 5$.

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This completed the proof of Fermat's last theorem, a full 358 years after Fermat wrote that little note in the margin.

Outline

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The geometry of all of these varieties was linked to certain number theoretic problems.

The study of these kinds of relationships is *arithmetic geometry*.

Thank You!

Questions?